Mathematical Techniques
for Engineers and Scientists

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Preface

Modern engineers and scientists are frequently faced with difficult mathematical problems to solve. As technology continues to move ahead, some of these problems will require a greater understanding of advanced mathematical concepts than ever before. Unfortunately, the mathematical training in many engineering and science undergraduate university programs ends with an introductory course in differential equations. Even in those engineering and science curriculums that require some mathematics beyond differential equations, the required advanced mathematics courses often do not make a clear connection between abstract mathematical concepts and practical engineering applications.

This mathematics book is designed as a self-study text for practicing engineers and scientists, and as a useful reference source to complement more comprehensive publications. In particular, the text might serve as a supplemental text for certain undergraduate or graduate mathematics courses designed primarily for engineers and/or scientists. It takes the reader from ordinary differential equations to more sophisticated mathematics—Fourier analysis, vector and tensor analysis, complex variables, partial differential equations, and random processes. The assumed formal training of the reader is at the undergraduate or beginning graduate level with possible extended experience on the job. We present the exposition in a way that is intended to bridge the gap between the formal education of the practitioner and his/her experience. The emphasis in this text is on the use of mathematical tools and techniques. In that regard it should be useful to those who have little or no experience in the subjects, but should also provide a useful review for readers with some background in the various topics.

Some special features of the text that may be of interest to readers include the following:

- Historical comments appear in a box at the beginning of many chapters to identify some of the major contributors to the subject.
- The most important equations in each section are enclosed in a box to help the reader identify key results.
- Boxes are also used to enclose important lists of identities and sometimes to summarize special results.
- Numbered examples are given in every chapter, each of which appears between horizontal lines.
- Exercise sets are included at the end of each chapter. Most of the problems in these exercise sets have answers provided.
- Remark boxes are occasionally introduced to provide some additional comments about a given point.
- At the end of each chapter is a "Suggested Reading" section which contains a brief list of textbooks that generally provide a deeper treatment of the mathematical concepts.
- A more comprehensive numbered set of references is also provided at the end of the text to which the reader is directed throughout the text, e.g., (see [10]).
- We have included a Symbols and Notation page for easy reference to some of the acronyms and special symbols as well as a list of Special Function notation (at the end of Chapter 2).

The text is composed of 15 chapters, each of which is presented independently of other chapters as much as possible. Thus, the particular ordering of the chapters is not necessarily crucial to the user with few exceptions. We begin Chapter 1 with a review of ordinary differential equations, concentrating on second-order linear equations. Equations of this type arise in simple mechanical oscillating systems and in the analysis of electric circuits. Special functions such as the gamma function, orthogonal polynomials, Bessel functions, and hypergeometric functions are introduced in Chapter 2. Our presentation also includes useful engineering functions like the step function, rectangle function, and delta (impulse) function. An introduction to matrix methods and linear vector spaces is presented in Chapter 3, the ideas of which are used repeatedly throughout the text. Chapters 4 and 5 are devoted to vector and tensor analysis, respectively. Vectors are used in the study of electromagnetic theory and to describe the motion of an object moving through space. Tensors are useful in studies of continuum mechanics like elasticity, and in describing various properties of anisotropic materials like crystals. In Chapters 6 and 7 we present a fairly detailed discussion of analytic functions of a complex variable. The Cauchy-Riemann equations are developed in Chapter 6 along with the mapping properties associated with analytic functions. The Laurent series representation of complex functions and the residue calculus presented in Chapter 7 are powerful tools that can be used in a variety of applications, such as the evaluation of nonelementary integrals associated with various integral transforms.

Fourier series and eigenvalue problems are discussed in Chapter 8, followed by an introduction to the Fourier transform in Chapter 9. Generally speaking, the Fourier series representation is useful in describing spectral properties of power signals, whereas the Fourier transform is used in the same fashion for energy signals. However, through the development of formal properties associated with the impulse function, the Fourier transform can also be used for power signals. Other integral transforms are discussed in Chapter 10— the Laplace transform associated with initial value problems, the Hankel transform for circularly symmetric functions, and the Mellin transform for more specialized applications. A brief discussion of discrete transforms ends this chapter. We present some of the classical problems associated with the calculus of variations in Chapter 11, including the famous brachistochrone problem which is similar to Fermat's principle for light. In Chapter 12 we give an introductory treatment of partial differential equations, concentrating primarily on the separation of variables method and transform methods applied to the heat equation, wave equation, and Laplace's equation. Basic probability theory is introduced in Chapter 13, followed by a similar treatment of random processes in Chapter 14. The theory of random processes is essential to the treatment of random noise as found, for example, in the study of statistical communication systems. Chapter 15 is a collection of applications that involve a number of the mathematical techniques introduced in the first 14 chapters. Some additional applications are also presented throughout the text in the various chapters.

In addition to the classical mathematical topics mentioned above, we also include a cursory introduction to some more specialized areas of mathematics that are of growing interest to engineers and scientists. These other topics include the fractional Fourier transform (Chapter 9), wavelets (Chapter 9), and the Walsh transform (Chapter 10).

Except for Chapter 15, each chapter is a condensed version of a subject ordinarily expanded to cover an entire textbook. Consequently, the material found here is necessarily less comprehensive, and also generally less formal (i.e., it is presented in somewhat of a tutorial style). We discuss the main ideas that we feel are essential to each chapter topic and try to relate the mathematical techniques to a variety of applications, many of which are commonly associated with electrical and optical engineering—e.g., communications, imaging, radar, antennas, and optics, among others. Nonetheless, we believe the general exposition and choice of topics should appeal to a wide audience of applied practitioners. Last, we wish to thank our reviewers Christopher Groves-Kirkby and Andrew Tescher for their careful review of the manuscript and helpful suggestions.

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2003

Symbols and Notation

a, **x**, **A**, ...	Vector, matrix, or random variable
AM	Amplitude modulation
arg, Arg	Argument
BC	Boundary condition
BVP	Boundary value problem
CDF	Cumulative distribution function
CNR	Carrier-to-noise ratio
Cov	Covariance
CTF	Coherent transfer function
CW	Continuous wave
DE	Differential equation
DFT	Discrete Fourier transform
DWT	Discrete Walsh transform
E[.]	Expectation operator
EO	Electro-optics
$\mathscr{F}\{.\}$	Fourier transform operator
FFT	Fast Fourier transform
FM	Frequency modulation
FRFT	Fractional Fourier transform
FWT	Fast Walsh transform
GRIN	Graded index
$\mathcal{H}\{.\}$	Hankel transform operator
IC	Initial condition
IF	Intermediate frequency
Im	Imaginary part of
ISP	Irregular singular point
IVP	Initial value problem
J	Jacobian
$\mathfrak{L}\{.\}$	Laplace transform operator
LO	Local oscillator
LRC	Inductor-resistor-capacitor network
LSI	Linear shift-invariant
M{.}	Mellin transform operator
Mod	Modulus
ODE	Ordinary differential equation
PDE	Partial differential equation
PDF	Probability density function
Pr	Probability
PV	Principal value
RC	Resistor-capacitor network
Re	Real part of
Res	Residue
RSP	Regular singular point
SNR	Signal-to-noise ratio
u_x, u_{xt}, ...	Partial derivative
U.H.P.	Upper half-plane
Var	Variance
z^*	Complex conjugate of z
δ_{jk}, δ_j^k, δ^{jk}	Kronecker delta
δy, δF	First variation
∇	Del (or gradient) operator
e_{ijk}, e^{ijk}	Permutation symbols
$\in$	Belonging to
$\notin$	Not belonging to
$\int_C$	Path or contour integral
$\oint_C$	Closed path (contour) integral
$\iint_S$	Surface integral
$\oiint_S$	Closed surface integtral
$\iiint_V$	Volume integral
$\cap$	Intersection
$<>$	Ensemble average
<**x**,**y**>	Inner product
$<a,b,c>$	Vector components

NOTE: Notation for special functions of mathematics is provided in Table 2.5 at the end of Chapter 2.

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