Uncertainty Quantification (UQ) Techniques to Improve Predictions of Laser Beam Control Performance

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ABSTRACT

Uncertainty quantification (UQ) is the study of the effects of uncertainty on the values of analytical results and the predictions of scientific models. Sources of uncertainty include imprecise knowledge of the exact values of parameters, lack of confidence in the physical models, use of imperfectly calibrated models, and irreducible uncertainties due to physical characteristics.

The Air Force Research Laboratory has undertaken the challenge of understanding, developing and analyzing the techniques of UQ as they apply to Laser Beam Control. This paper proposes a simple methodology and simple results with our first attempt of applying UQ as a new analysis tool. The software toolkit which was chosen was an analytical group of algorithms from a Sandia National Laboratory (SNL) package called DAKOTA (Design Analysis Kit for Optimization and Terascale Applications). The specific application of interest to the Air Force Research Laboratory (AFRL) is the analytical prediction of the performance of a Laser Beam Control systems under various scenarios, conditions, and missions. The application of rigorous UQ techniques to the models used to predict beam control performance could greatly improve our confidence in these predictions and also improve the acceptance of advanced Laser Beam Control systems within the science and engineering communities[1,2]. The proposed work would follow a multi-step approach, analyzing the more easily quantified sources of uncertainty, and then including increasingly complicated physical phenomena as the work progresses. Will present the initial results, and the first steps in the incorporation of UQ into our Laser Beam Control Modeling and Simulation environments.

KEYWORDS: Uncertainty quantification, DAKOTA, Laser Beam Control, Atmospheric Turbulence, Atmospheric Characterization, Statistics, Correlation Techniques.

1. INTRODUCTION

The U. S. Air Force Research Laboratory, Directed Energy Directorate (AFRL/RD), as well as others, are interested in uncertainties associated with Modeling and Simulation or any analysis as applied to many laser beam control systems. The particular application of interest to the AFRL is the analytical prediction of the performance of laser systems (LS) under various scenarios, conditions, and missions, and provide error bounds on the results. Thus, the hope is the application of rigorous UQ techniques to the models used to predict laser performance could greatly improve our confidence in these predictions.

There are a variety of sources which would give rise to uncertainty when modeling Laser Beam Control systems[3]. A short list is the propagation of the laser beam through the atmosphere, the modeling of the material properties under stressing environments, structural, and associated with structural are aero effects like aero optics and aero mechanical, when the laser system is aboard and aircraft.

The work presented will follow a multi-step approach, analyzing the more easily quantified sources of uncertainty first, and then including increasingly complicated physical phenomena as the work progresses. The first step in this application was to fully understand Uncertainty Quantification (UQ). Thus, the first

portion of this article deals which the explanation and description of UQ. Then, the decision to see if UQ can be applied to the improvement in the models, simulations and predictions of laser beam control estimates. Therefore, a problem statement of the laser beam control is developed. Finally, a methodology was established and seen if it can be successfully applied to the problem statement. Will present the initial results, and the first steps in the incorporation of UQ into our beam control modeling and simulation environments.

2. DESCRIPTION OF UNCERTAINTY QUANTIFICATION

What is Uncertainty Quantification (UQ)? In our research we have found that there are many definitions of UQ and that researcher's use of UQ is varied. A general definition is that UQ is the study of the effects of uncertainty on the values of analytical results and the predictions of scientific models. Sources of uncertainty include imprecise knowledge of the exact values of parameters, lack of confidence in the physical models, use of imperfectly calibrated models, and irreducible uncertainties due to physical characteristics. And more precisely, for this article, UQ will help us study the effects of uncertainty on Laser Beam Control values and analytical results and the predictions of our scientific models, used in modeling of Laser Beam Control systems. Sources of uncertainty include imprecise knowledge of the exact values of parameters, lack of confidence in the physical models, use of imperfectly calibrated models, and irreducible uncertainties due to physical characteristics.

An application of UQ is for Decision Making[4,5]. In spite of the wide spread use of Modeling and Simulation (M&S) tools it remains difficult to provide objective confidence levels in the quantitative information obtained from numerical predictions. The complexity arises from the uncertainties related to the inputs of any computation attempting to represent a real physical system. Use of M&S predictions in high-impact decisions require a rigorous evaluation of the confidence levels of the input data.
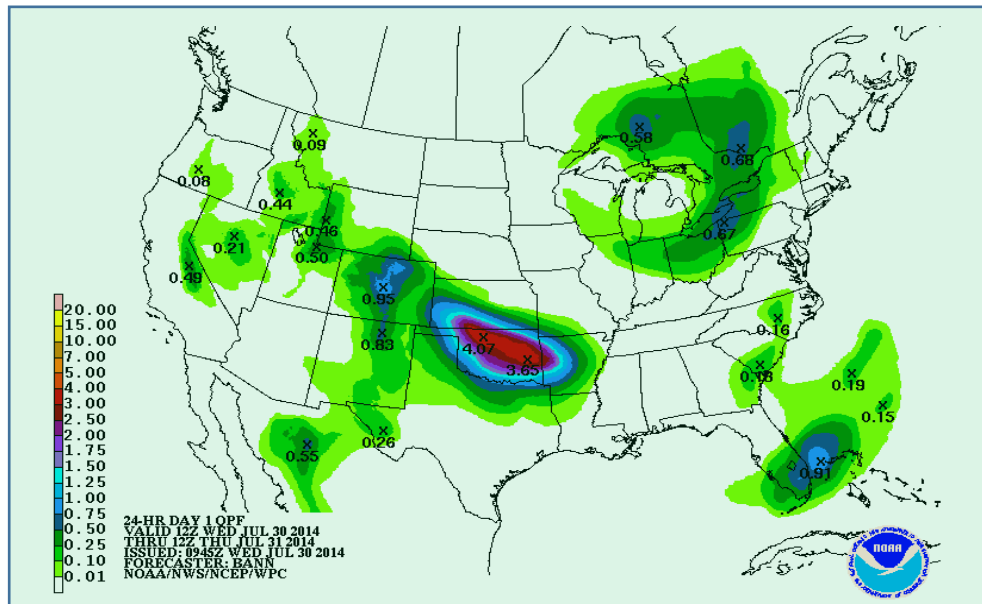

Figure 1: Example: Weather Prediction

Another application of UQ is for simply finding out what the error bars are on numerical results for a given simulation or experiment. Sources of error are everywhere in the natural sciences and engineering. The error sources could also be associated with the measurement equipment, but mostly for this article, modeling the general unpredictable behavior associated with the atmosphere. This atmospheric error can be compounded if the laser is put on an aircraft and aero-effects like vibration and aero sheer layer have to be taken into consideration. The results provide an intuitive notion of confidence. One important objective of UQ is to make the intuitive notion of confidence mathematically sound! This is the way UQ was utilized in this article.
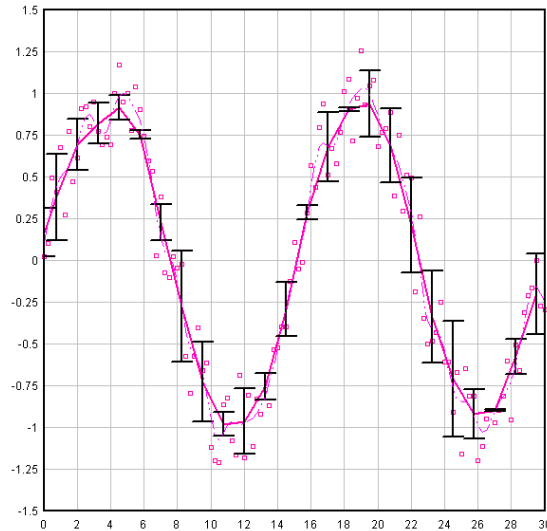
Figure 2: Example: Generic data with error Bars

In addition UQ can be classified as Aleatory or Epistemic[4,5]. Aleatory UQ can be defined if the physical variability is present in the system or its environment. It is not strictly due to a lack of knowledge and cannot be reduced (also referred to as variability, stochastic uncertainty or irreducible uncertainty). It is naturally defined in a probabilistic framework. Examples are: material properties, operating conditions manufacturing tolerances, etc. In mathematical modeling it is also studied as noise. Aleatory uncertainty cannot be reduced as it arises naturally from observations of the system. Additional experiments can only be used to better characterize the variability.

Epistemic, as the name inplies, is a potential deficiency that is due to a lack of knowledge[4,5]. It can arise from assumptions introduced in the derivation of the mathematical model (it is also called reducible uncertainty or incertitude). Examples are: Aero-optical effects at high Mach number, which has never been measured or characterize, turbulence model assumptions or surrogate chemical models which have never been studied. It is NOT naturally defined in a probabilistic framework. Can lead to strong bias of the predictions. As already mentioned, epistemic uncertainty can be reduced by increasing our knowledge, e.g. performing more experimental investigations and/or developing new and better physical models.

One more observation about UQ. UQ is often confused, misrepresented or misdefined as Sensitivity Analysis. Sensitivity Analysis (SA) investigates the connection between inputs and outputs of a (computational) model. The objective of SA is to identify how the variability in an output quantity of interest (q) is connected to an input (x) in the model; the result is a sensitivity derivative dq/dx. SA allows one to build a *ranking* of the input sources which might dominate the response of the system. Note that strong, large sensitivities derivatives do not necessarily translate in critical uncertainties because the input variability might be very small in a specific device of interest.

3. LASER BEAM CONTROL PROBLEM STATEMENT

The Laser Beam Control section is an essential part of any laser system. The Laser Beam Control section is the section where the majority of the optics, telescope, adaptive optics (AO) and atmospheric correction occur. It is the uncertainties which arise from the atmosphere and the associated correction which is studied in this article.
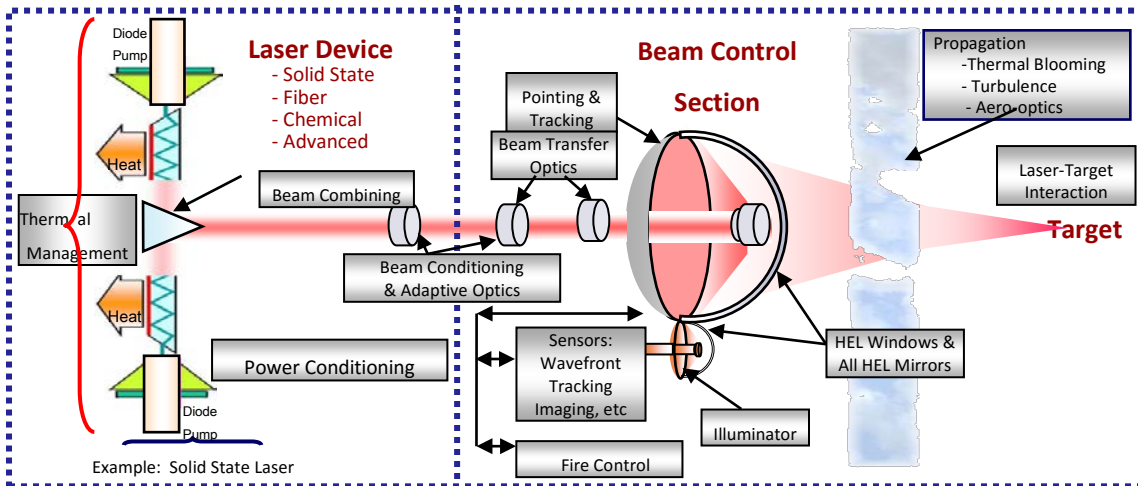
Figure 3: Laser Section and Laser Beam Control Section

We have mathematical models for all these components, both for the Laser Section and the Laser Beam Control section. In the Laser Beam Control section, the atmospheric input parameters to the model whose exact values are random and cannot be controlled in physical experiments will always occur. The application for UQ in this article comes in attempting to model the atmosphere, since as stated, the atmospheric parameters are random in nature.

Thus, questions always naturally arise, like how accurately does our mathematical models describe the true physics of all the components and the atmosphere, and what is the impact of model uncertainty (structural or parametric) on outputs from the model? Also, what are the error bars in any simulation which contains any modeling of the atmosphere? Given a mathematical model, how accurately is a specified output approximated by a given numerical method? Or given a mathematical model and numerical method, the error in numerical solutions and specified outputs be reliably estimated and controlled by adapting resources?

4. PROPOSED METHODOLOGY

Since UQ has never been applied to Laser Beam Control modeling and simulation, a new paradigm had to be developed to study this application. It was decided to use the Error Tree analysis, which is developed using the program called, Scaling for HEL and Relay Engagement (SHaRE), developed for the Federal Government by MZA. This program SHaRE does theoretical analysis of laser propagation and beam control performance[9]. The analytical methods used are the basis of system performance scaling laws that model the anticipated performance of a laser projection, imaging, adaptive-optics system, and atmospheric propagation, given diverse laser, environment, and engagement conditions. These methods are fast-running compared to wave-optics simulations by providing a broader-ranging assessment of the laser system at lower fidelity.

With this Error Tree analysis we can track the error bars as we propagate them though the simulation. The next point is to actually develop the steps of the methodology.
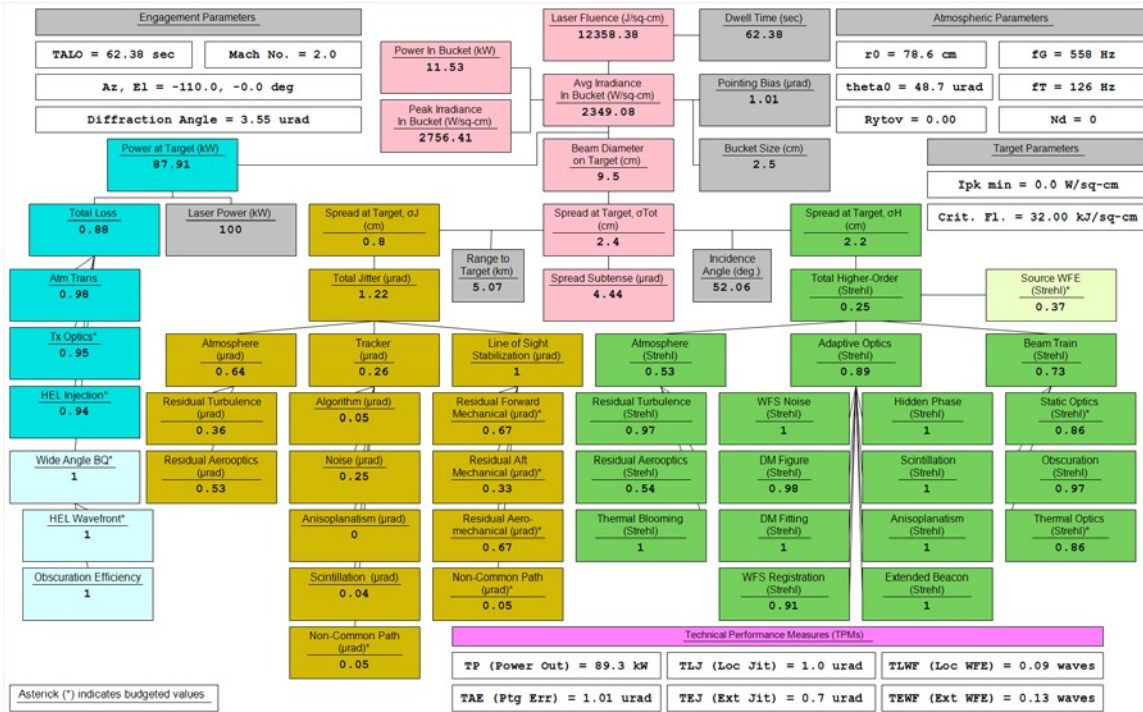
Figure 4: Error Tree from the program SHaRE

The new paradigm or methodology that was developed is shown next.

Step 1 – Use Error Tree from Scaling Law: This first step with using the scaling law program SHaRE and the error tree was already mentioned and developed above[9].

Step 2 – Select model to be simulated: The MATRIX is the Laser System testbed which was chosen to model. This testbed; Mobile Active Targeting Resource for Integrated Experiments (MATRIX), has been extensively documented and the following references are given[6,7]. But briefly, this Laser System testbed has a complete Laser Beam Control system which can be used in this simulation. It has a 30 cm Beam Director, with 100 °/sec, 50 °/sec^2 Agility and Gyro Stabilization. It has Precision Pointing, Fast Steering Mirror (FSM's), High Bandwidth Tracking, with less than 3 mirco-radians demonstrated. Although not shown in the diagram, nor initially used in the simulation, the MATRIX testbed does have the capability to insert a complete Adaptive Optics subsystem with Deformable Mirror and appropriate Wave Front Sensor. The MATRIX testbed has Multi-Sensor Acquisition and Tracking, with MWIR, NIR, Visible at 500 Hz Frame Rates and Laser Ranging. It has the ability to insert a variety of surrogate lasers for laser engagement and In-Situ scoring. It is show below. In Figure 5. This Laser System has already been extensively modeled and in addition, there is plenty of data which is available to be used to apply the UQ process.

Step 3 – Select model inputs and outputs desired: It is worth mentioning that all inputs and outputs were already available in the existing model for the MATRIX testbed, and used for this simulation. The inputs and outputs desired are derived solely from the atmospheric data. This is the data which was most abundant and also mostly unpredictable and thus, perfect for the application of UQ. The inputs were Input-1: atmospheric turbulence strength C_n^2, (pronounced C-N squared) which is the measure of the atmospheric turbulence strength. This data was taken at the 2-mile laser test site at the Starfire Optical Range (SOR) at Kirtland AFB. Although this quantity is all measure data, the equation of C_n^2 is shown here for

completeness, and the units are (m$^{-2/3}$)[8,9]. The variables are P, the mean temperature (hPa), T is the mean temperature in (deg K), and the C_T^2 is the temperature structure constant.

$$C_N^{\ 2} = C_T^{\ 2}\left[79\times10^{-6}\frac{P}{T^2}\right]^2$$

(1)

The second input is Input-2: Wind speed. Wind speed was taken simultaneously with an anemometer, also from the 2-mile laser test site at the SOR at Kirtland AFB. In future analysis, more inputs could also be consisted, such as Temperature, Humidity, others and SME desired inputs. For now these two inputs should suffice.

The desired outputs selected from the simulation are:
- Output 1 - Power In the Bucket (PIB), this variable is called "Integral" in the simulation, and is the summed power within a prescribed diameter, like a 5 centimeter diameter. Usually in the far-field, i.e., at the target. Its units are just watts[8,9]. If we consider a circular PIB, this is the equation:

$$PIB(\theta) = \int_0^{2\pi}\int_0^{\theta} I\ (\phi,\varphi)\phi d\phi d\varphi = 2\pi\int_0^{\theta} I_{ave}(\phi)\phi d\phi$$

(2)

Where $I(\phi,\varphi)$ is any far-field intensity distribution and ϕ is the tangential coordinate.

- Output 2 - Peak Irradiance on target, called "PeakIrrad" in the simulation, and is the peak power of the far field pattern on the target. Usually the highest pixel value at the far field pattern[8,9]. Its units are watts/m^2. Now knowing the P, power at the far field, the A, aperture area, and the R, range and λ, the wavelength. The equation is :

$$I_{peak} = \frac{PA}{\lambda^2 R^2}$$

(3)

- Output 3 - Average Irradiance on target, this variable is called "AvgIrrad" in the simulation, and is the average power of the far field pattern on the target. Its units are watts/m^2. The equation is[8,9];

$$I_{Avg}(r_a) = \frac{1}{\pi r_A^2}\int_0^{2\pi}\int_0^{r_a} I_{peak}e^{\frac{-r^2}{2\sigma^2}} r dr d\theta$$

(4)

Where r_a is the radius to be included in the average calculation, and σ is the standard deviation of the beam on target. These three irradiance values are functions of the geometry and system parameters. The rest of the outputs are all functions of the atmospheric path and specifically the atmospheric turbulence strength, C_n^2.

- Output 4 - Coherence diameter of the atmospheric path, this variable is called "r_o" in the simulation, also called the Fried diameter or Fried's coherence length, after David Fried[8,9]. Basically, just look at this parameter as the average diameter of the atmospheric turbulence. Its units are meters or centimeters. The following is the equation:

$$r_0 = \left[0.42\sec(\zeta)k^2\int_0^L C_N^{\ 2}(h)dh\right]^{-3/5}$$

(5)

Where ζ is the zenith angle, $k \equiv$ wavenumber $= 2\pi/\lambda$, h is the height above the ground, L is the length of the turbulent regime, and C_n^2 is the atmospheric turbulence structure function.

- Output 5 - Rytov parameter along atmospheric path, this variable is called "Rytov" in the simulation, a scaling parameter for laser propagation through horizontal atmospheric turbulence[8,9]. Though strictly a theoretical quantity, the Rytov parameter is a useful metric of the optical effects

for extended-turbulence propagation and is a leading indicator of the performance limitations of adaptive-optical compensation devices. Also, Rytov numbers greater than 0.2 are generally considered to be strong <u>scintillation</u>. The Rytov number is designated as σ_x^2 and the equation is;

$$\sigma_x{}^2 = 0.5631\left(\frac{2\pi}{\lambda}\right)^{7/6} \int_0^L C_N{}^2(z)\left[z\left(\frac{1-z}{L}\right)\right]^{5/6} dz \tag{6}$$

- Output 6 - Greenwood Frequency, this variable is called "f_G", in the simulation. This is the frequency or bandwidth required for optimal correction with an adaptive optics system. Its units are hertz. The equation is[8,9];

$$f_G = \left[0.102k^2 \int_{path} C_N{}^2(h(z))V^{5/3}(z)dz\right]^{3/5} \tag{7}$$

Where V is the effective wind velocity.

- Output7 - Tyler Frequency, this variable is called "f_T", in the simulation[8,9]. This is the frequency or bandwidth required for optimal correction of x-tilt and y-tilt in an adaptive optics system. Its units are hertz. The equation is;

$$f_T = 0.368D^{-1/6}\lambda^{-1}\sqrt{\left[\int_{path} C_N{}^2(h(z))[V(z)]^2 dz\right]} \tag{8}$$

Where D is the diameter of the telescope aperture.

Step 4 – Utilized data when available to establish statistical variations. As mentioned above, the MATRIX testbed has been modeled in great detail, and in addition there is a variety of data available to utilize. This included both data associated with the actual testbed as well as the atmospheric data for the laser range with the MATRIX is used at. This atmospheric data was highly utilized to perform this UQ analysis[9].

Step 5 – Where data does not exit, utilize Subject Matter Experts (SME) appropriately. There are situations where data may not be available, may be scarce, or may be suspect. These are the situations where a Subject Matter Expert (SME) could be of great help. The SME could substitute some appropriate data from a similar experiment, which could be utilized for a particular simulation. Or the SME could spot some inappropriate results form a particular data set and warn the modeling and simulation team. And a variety of other possible scenarios.

Step 6 – Select DAKOTA as the Uncertainty Quantification environment. This set of powerful simulation toolkits was developed by Sandia National Laboratories (SNL). Not only was these toolkits highly recommended, but it was easy to have personal contact and ask questions from the developer, when needed in utilizing these programs, since SNL is on the same location as AFRL. The <u>D</u>esign <u>A</u>nalysis <u>K</u>it for <u>O</u>ptimization and <u>T</u>erascale <u>A</u>pplications, (DAKOTA) toolkit, is a general-purpose software toolkit for performing systems analysis and design on high performance computers[10]. Dakota provides algorithms for design optimization, uncertainty quantification, parameter estimation, design of experiments, and sensitivity analysis, as well as a range of parallel computing and simulation interfacing services.

Step 7 – Propagate these variations through the Error Tree. The results of this step are seen on the last section of this article.
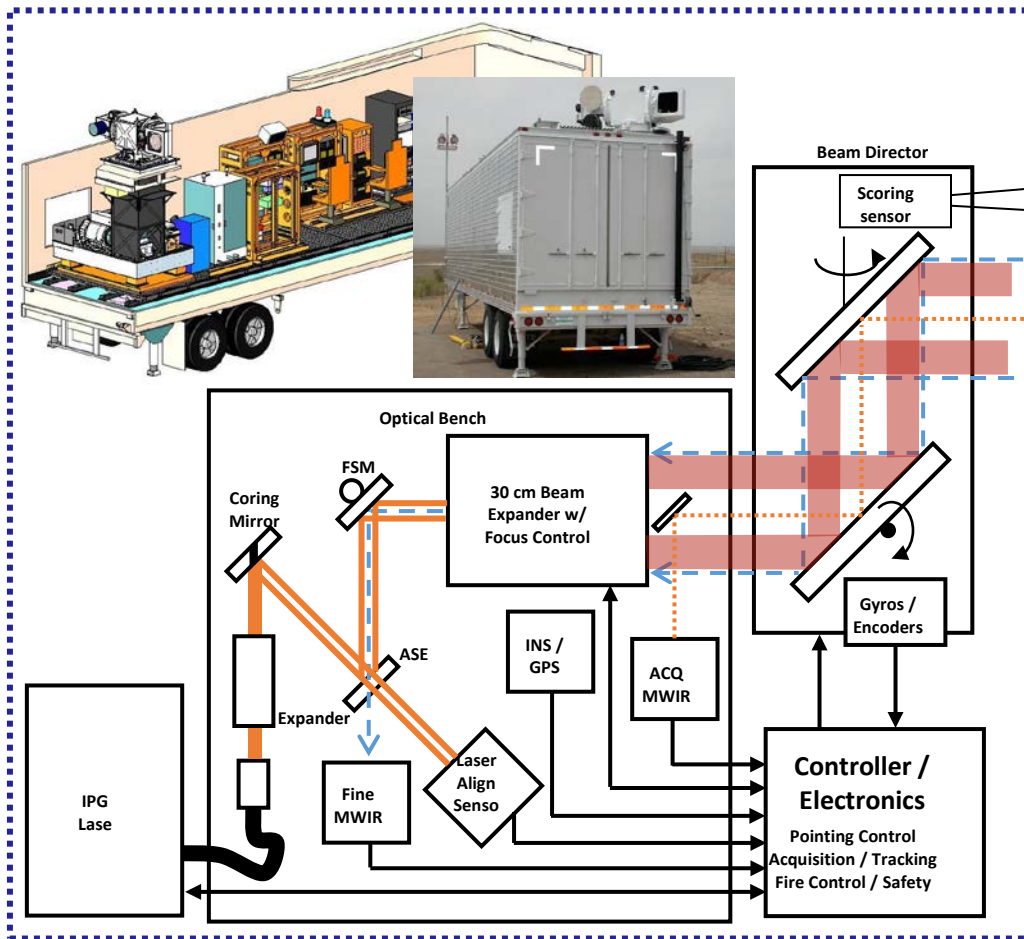
Figure 5: Mobile Active Targeting Resource for Integrated Experiments (MATRIX)

5. APPLICATION of DAKOTA to SHaRE

Most of the analysis will be performed by SHaRE, since the model of the MATRIX Testbed is already modeled there, especially the error tree analysis. However, there has to be an interface of sorts from SHaRE to DAKOTA. In order to do this an Application Program Interface (API) was written in "Perl". Perl is a family of high-level, general-purpose, interpreted, dynamic programming, general-purpose Unix scripting language. This API could have just as easily been written in another programming language, like C++. Figure 6, shows an interface diagram of this effort.

Thus, this API program allows calls to code base under SHaRE with the DAKOTA toolset. It translates code output for inputs to DAKOTA, and formats the data under a results file. In addition, to translates DAKOTA output parameters into code of inputs back into SHaRE. An additional comment is that the DAKOTA toolset runs on LINUS based computers. And is usually meant to run on a Supercomputer. For our application, we run the DAKOTA toolset and the SHaRE program on a stand-alone LINUS computer.
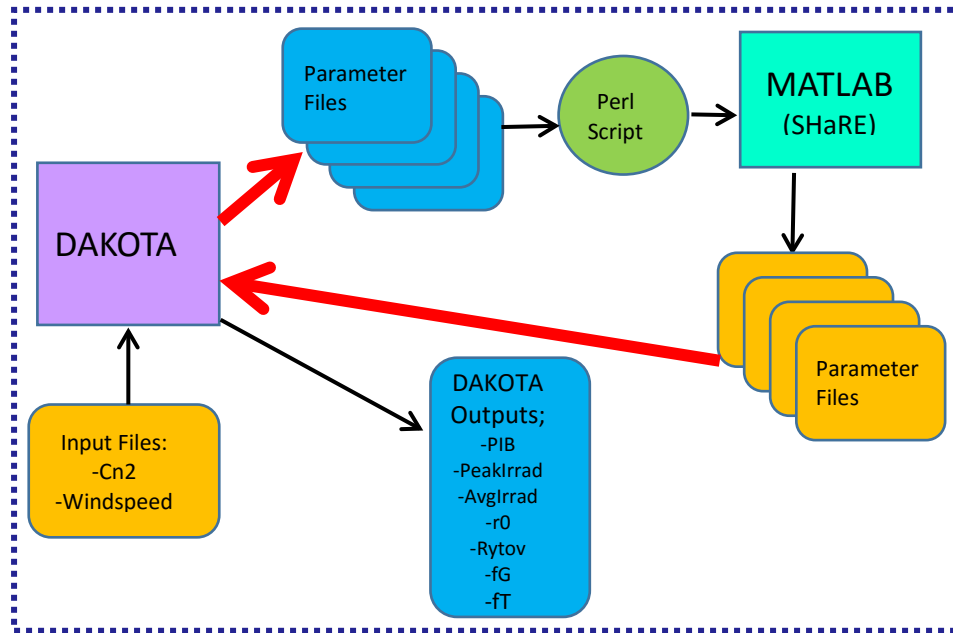
Figure 6: Software Interfaces for data flow

6. UQ ANALYTICAL ANALYSIS

Once all the correct software interfaces are validated, the modeling and simulation can commence. The analysis starts with the selection of the data which is given to the software model. Recall the inputs for this model were the atmospheric turbulence strength; C_n^2, and the wind speed. The inputs to the model were from a data gathering campaign which was done at the Starfire Optical Range (SOR). The instrument for the Cn2 was a Turbulence Profiler. The following figure shows the results which were produce by this measuring instrument.
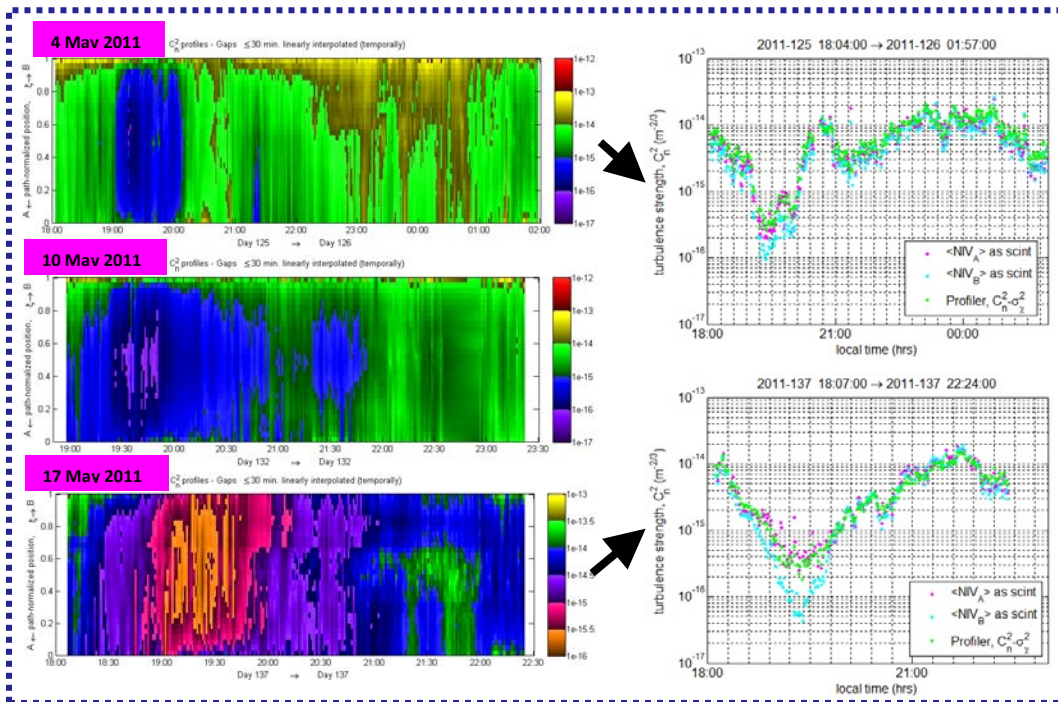

Figure 7. Atmospheric turbulence strength; Cn2 Profiles

The color "waterfall" plots on the left, show the output of the Turbulence Profiler instrument, with the C_n^2 as a function of path length and time of day. The horizontal axis is the strip of data taken at a given time, versus the distance (on the vertical axis), from the start of the path to the end of the path. From this plethora of data, one can average out the results to get a more recognizable chart of C_n^2, on the right side of the figure. Figure 7, shows plots for measurement results for three day. There was data for many days taken in the four season of the year. Some of that data was from very bad days (i.e. very rainy, windy, or otherwise undesirable) and others for very good days. This is where in Step 4., "Utilize Subject Matter Experts (SME)" would be appropriately applied.

During this same period of time Anemometer-Wind speed data was also recorded from a small weather station at the SOR. In addition the Turbulence Profiler instrument, also has the ability to extract wind speed from its measurements. Both were compared and contrasted. Below is the data from both instruments.
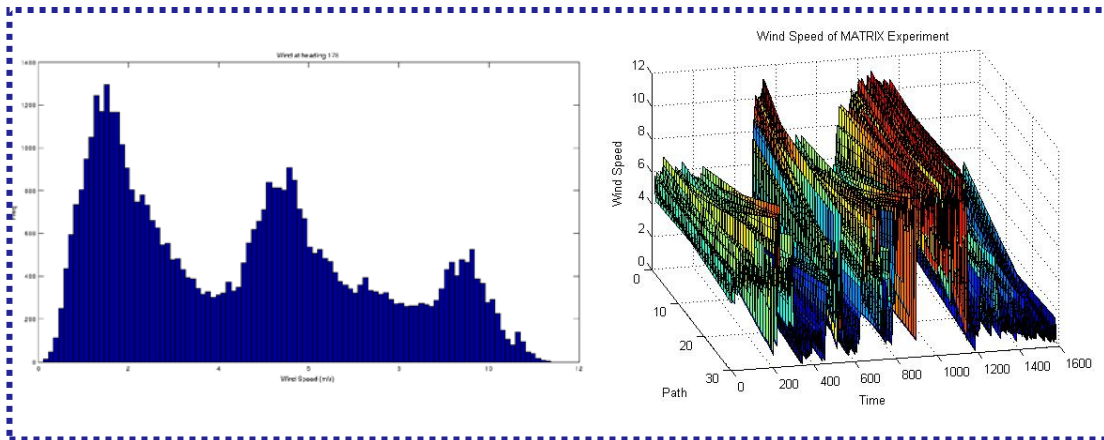

Figure 8. Wind Speed from Anemometer (left) and the Turbulence Profiler (right)

7. UQ ANALYTICAL RESULTS

Now with the input data set identified, we have the ability to set the entire simulation into operation. For this simulation run, a 10 watt laser at 2 miles was simulated. The first output from DAKOTA was very long and almost undecipherable. The next figure shown displays a very small portion of the output results.
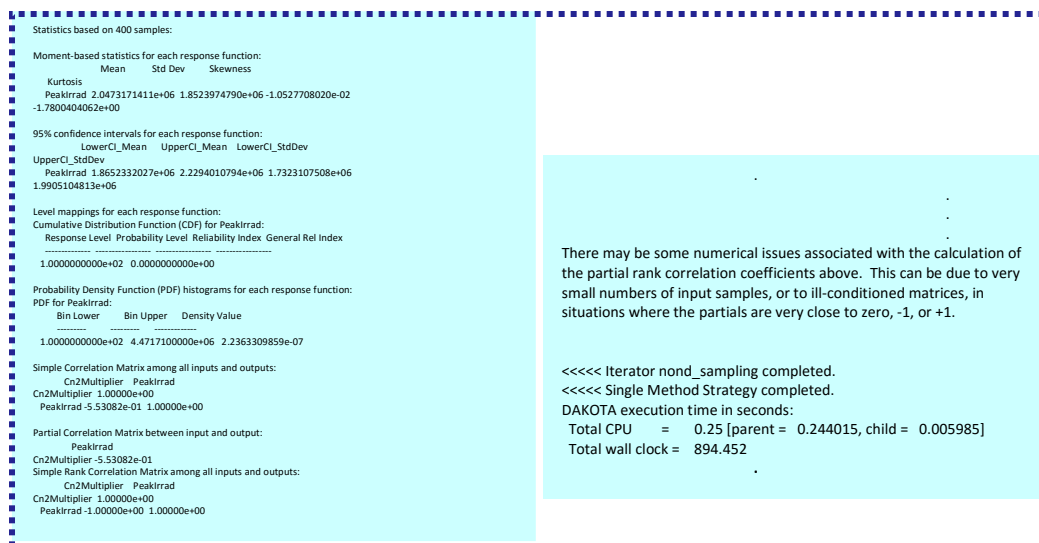

Figure 9. Actual output from a DAKOTA software run

It took some effort to "study and analyze" and "data mine" from these prolific DAKOTA output results to finally be able to make sense of the results. Looking at the results below, after 3000 iterations, there were some results "extracted" for our analysis. Shown are some statistical results for the DAKOTA output. But in addition, DAKOTA also calculated, many other values, such as, correlation, partial correlation, cross-correlation, autocorrelation, rank, etc., etc., between and among output values, and more. Only some Moment-based statistics and some correlations are shown below.

		Wind Speed (m/s)	$\log_{10}(C_n^2)$	Integral	Peak Irradiance M	Average Irradiance M	r_0 cm	Rytov	G	fT
Distribution Statistics	mean	4.575	-14.254	2.401	1.597	1.232	7.478	0.234	76.75	10.71
	Std	2.852	0.423	1.053	0.807	0.54	4.748	0.281	83.32	10.36
	Skewness			-0.374	-0.125	-0.374	2.029	5.627	2.849	2.22
	Kurtosis			-1.065	-1.2	-1.065	8.071	72.36	16.47	9.292
Correlation	WindSpeed	1	0.545	-0.356	-0.357	-0.356	-0.501	0.421	0.741	0.791
	Cn2exp	0.545	1	-0.88	-0.891	-0.88	-0.922	0.784	0.765	0.77
	Integral	-0.356	-0.88	1	0.995	1	0.71	-0.75	-0.696	-0.688
	PeakIrrad	-0.357	-0.891	0.995	1	0.995	0.744	-0.719	-0.672	-0.668
	Avg	-0.356	-0.88	1	0.995	1	0.71	-0.75	-0.696	-0.688
	r0	-0.501	-0.922	0.71	0.744	0.71	1	-0.565	-0.576	-0.594
	Rytov	0.421	0.784	-0.75	-0.719	-0.75	-0.565	1	0.858	0.82
	G	0.741	0.765	-0.696	-0.672	-0.696	-0.576	0.858	1	0.996
	fT	0.791	0.77	-0.688	-0.668	-0.688	-0.594	0.82	0.996	1
Partial Correlation	WindSpeed			0.309	0.338	0.309	0.004	-0.011	0.601	0.694
	Cn2			-0.875	-0.889	-0.875	-0.894	0.73	0.642	0.662
Rank Correlation	WindSpeed	1	0.533	-0.332	-0.342	-0.332	-0.533	0.533	0.885	0.905
	Cn2exp	0.533	1	-0.909	-0.917	-0.909	-1	1	0.836	0.811
	Integral	-0.332	-0.909	1	1	1	0.909	-0.909	-0.653	-0.625
	PeakIrrad	-0.342	-0.917	1	1	1	0.917	-0.917	-0.664	-0.636
	Avg	-0.332	-0.909	1	1	1	0.909	-0.909	-0.653	-0.625
	r0	-0.533	-1	0.909	0.917	0.909	1	-1	-0.836	-0.811
	Rytov	0.533	1	-0.909	-0.917	-0.909	-1	1	0.836	0.811
	G	0.885	0.836	-0.653	-0.664	-0.653	-0.836	0.836	1	0.999
	fT	0.905	0.811	-0.625	-0.636	-0.625	-0.811	0.811	0.999	1
Partial Rank Correlation	WindSpeed			0.43	0.434	0.43	-0.009	-0.01	0.947	0.955
	Cn2			-0.917	-0.924	-0.917	-1	1	0.925	0.913

Figure 10. Rearranged output from a DAKOTA software run

As one can see this DAKOTA gives Moment-based statistics for each output variable. For example in the "PIB (or Integral)", we get 2.401watts in a 10 centimeter diameter stop, i.e., the "bucket size", with a standard deviation of 1.053watts. Skewness and Kurtosis are just the statistical definitions found in any higher level statistical text[11].

8. CONCLUSIONS

The interface of SHaRE and DAKOTA, and specifically Uncertainty Quantification was indeed useful for determining the uncertainty on the values of analytical results. The predictions add insight and understanding of the Laser Beam Control system results from our models. Sources of uncertainty studied here included imprecise knowledge of the exact values of parameters associated with the laser propagation through the atmosphere. Specifically the atmospheric turbulence strength; C_n^2, and the wind speed. These input files were inserted into the developed interface simulation which comprised of SHaRE and DAKOTA. After 3000 iterations, statistical results were obtained, in which we could quantify the uncertainty in our selected outputs, by looking at the mean values and their standard deviations.

In the near future it is planned to switch over to a more accurate, but more time consuming, Wave Propagation program (i.e. WaveTrain) as opposed to a less accurate scaling law code like SHaRE. Since a Wave Propagation program, and DAKOTA for that matter, are very large and time consuming software programs, it is intend to migrate to more powerful super computer environments. Also, since DAKOTA is such a large, and all in composing program, it should be used and studied farther. DAKOTA's UQ toolbox should be studied more to understand more of its capability. Other DAKOTA toolboxes like the Optimization, Parameter Estimation and Sensitivity Analysis toolboxes should also be incorporated into further studies.

ACKNOWLEDGMENTS

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