Energy requirements for quantum computation

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ABSTRACT

A lower bound on the amount of energy needed to carry out an elementary logical operation on a qubit system, with a given accuracy and in a given time, has been recently postulated. This paper is an attempt to formalize this bound and explore the conditions under which it may be expected to hold. For a specific, important case (namely, when the control system is a quantized electromagnetic field) it is shown how one can extend this result to a generally stronger constraint on the minimum energy *density* required, per pulse.

Keywords: quantum computation, quantum noise, uncertainty principle

1. INTRODUCTION

It has become of interest lately to explore the constraints that the quantum nature of the control degrees of freedom might impose on the practical operation of quantum logical gates.[1–4] A very general result derived recently by Ozawa[4] is that any quantum gate that changes the energy or angular momentum state of a qubit will require a minumum number of ancillary bosons of the order of $1/\epsilon$, if it is to have a failure probability smaller than ϵ. If the bosons are excitations of a quantum harmonic oscillator (such as, e.g., photons) of frequency ω, this becomes a minimum energy requirement

$$E_{min} \sim \frac{\hbar\omega}{\epsilon} \tag{1}$$

in agreement with previous studies[1,3] which focused on the effect of the quantum nature of the electromagnetic field on the performance of logical gates.

Ozawa's result has very wide applicability, but it must be kept in mind that it is relatively straightforward (and it may be, in fact, advantageous for other practical reasons) to encode a logical qubit in degenerate states of systems of a few qubits, which are mutually interconvertible without any energy or angular momentum cost: for instance, the encoding in a 3-qubit decoherence-free subsystem[5,6] uses as the logical zero the state $|0\rangle_L = 2^{-1/2}(|01\rangle - |10\rangle)|0\rangle$ of three physical qubits, and as the logical one the state $|1\rangle_L = 6^{-1/2}(|100\rangle + |010\rangle - 2|001\rangle)$. These two states have the same quantum numbers for total angular momentum and energy; in fact, they simply represent the two different ways to get a state with $l = 1/2$ and $m = -1/2$ in a system of three spin-1/2 particles. For such an encoding, conservation of total energy or angular momentum alone does not appear to restrict the possible logical operations.

I have recently shown[7] that in many cases, regardless of whether a conservation law is broken or not by the action of the logical gate, there is a minimum requirement on the energy of the "control" system, or degree of freedom, of the form (1) if the system is an oscillator, or more generally of the form

$$E_{min} \sim \frac{\hbar}{\epsilon T} \tag{2}$$

if the gate is to be carried out in a time T with failure probability less than ϵ. My analysis covers gates mediated by external electromagnetic fields, or by controlled collisions between particles, assuming that the fields or particles are in minimum uncertainty "coherent states." There are, nonetheless, some questions still open, regarding the full generality of the result, and, for instance, whether placing the control degree of freedom in a nonclassical state (such as a squeezed state) might lower the bounds or not. In this paper I shall attempt to express the constraint (2) as a formal postulate, and exhibit a number of worked out examples and ideas for how a general proof might proceed. I will also show how for a large class of systems (atom-like qubits interacting

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Noise and Information in Nanoelectronics, Sensors, and Standards, L. B. Kish, F. Green, G. Iannaccone, J. R. Vig, Editors, Proceedings of SPIE Vol. 5115 (2003)

with a quantized electromagnetic field) the constraint (2) can, in general, be strengthened, resulting in a more precise specification of the volume of space where the minimum energy (2) should be located, or equivalently, in a minimum *energy density* requirement. Interestingly, this result turns out to be equivalent, for these systems, to the familiar requirement[8] that the probability of spontaneous emission should be negligible during the operation of the gate.

This paper is organized as follows. In the next section, the formal postulate is presented, along with some heuristic arguments for its validity. Section 3 deals with several worked-out examples. Section 4 presents the new results for atom-like qubits. Finally, Section 5 has some discussion and conclusions.

2. A FORMAL POSTULATE

2.1. Using a quantum control to achieve a conditional sign flip

In order to focus only on the constraints arising from the quantum nature of the control, and not on those imposed by conservation laws, I consider here only a particular kind of two-qubit gate which preserves the qubits' energy and angular momentum (assuming the $|0\rangle$ and $|1\rangle$ states are eigenstates of these variables), namely, the controlled sign-flip gate, which leaves the states $|00\rangle, |01\rangle$ and $|10\rangle$ unchanged but turns $|11\rangle$ into $-|11\rangle$. The role of the control system is, essentially, to switch "on" and "off" a Hamiltonian which accomplishes this in a time T, to an accuracy given by ϵ.

To that end, let the control degree of freedom be initially in the state $|\psi_0\rangle$, and let its self-Hamiltonian be H_0. Let the interaction Hamiltonian have the simple form $H_I = V|11\rangle\langle 11|$. This is the minimal form needed for the purpose at hand, and V need depend only on "control" operators. Further, suppose that V is time-independent in the Schrödinger picture, although this may not be necessary. What *is* necessary is that the interaction be turned on and off only by the control system, acting under the influence of its own self-Hamiltonian (it follows that $|\psi_0\rangle$ cannot be a stationary state). Formally, we require

$$\langle\psi_0|V^2|\psi_0\rangle \simeq \langle\psi_0|e^{\frac{i}{\hbar}\int_0^T H_0\,dt'}|V^2|e^{-\frac{i}{\hbar}\int_0^T H_0\,dt'}|\psi_0\rangle \simeq 0 \tag{3}$$

at the initial and final times, $t = 0, T$. The reason to use V^2, and not just V, in (3) is that one could have situations where $\langle\psi_0|V|\psi_0\rangle = 0$ due to some symmetry, without necessarily implying that the interaction is "off" at that time. For instance, if the system is a harmonic oscillator, with $V \sim a$ (the annihilation operator) and $|\psi_0\rangle = |n\rangle$, one has $\langle\psi_0|V|\psi_0\rangle = 0$, yet such a V would have a large effect on the state $|\psi_0\rangle$.

To capture the desired change in sign at the end of the time T, we define the "failure probability" of the gate by considering what it does to a state such as $|00\rangle + |11\rangle$. What we want is something like

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|\psi_0\rangle \to \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)e^{-\frac{i}{\hbar}\int_0^T H_0\,dt'}|\psi_0\rangle \tag{4}$$

with the control system completely factorizing out of the part of the state that represents the two qubits. In other words, we want, among other things, that, by the end of the operation, the qubits leave no trace on the control system of what state they were in originally. What we will really get, however, is instead,

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|\psi_0\rangle \to \frac{1}{\sqrt{2}}|00\rangle e^{-\frac{i}{\hbar}\int_0^T H_0\,dt'}|\psi_0\rangle + \frac{1}{\sqrt{2}}|11\rangle e^{-\frac{i}{\hbar}\int_0^T (H_0+V)\,dt'}|\psi_0\rangle \tag{5}$$

which shows that the qubits and the control system are generally left in an entangled state. The "failure probability" can now be defined as 1 minus the square of the overlap between (4) and (5), i.e.,

$$\begin{aligned} p \quad &= 1 - \frac{1}{4}\left|1 - \langle\psi_0|e^{\frac{i}{\hbar}\int_0^T H_0\,dt'}e^{-\frac{i}{\hbar}\int_0^T (H_0+V)\,dt'}|\psi_0\rangle\right|^2 \\ &= 1 - \frac{1}{4}\left|1 - \langle\psi_0|\mathcal{T}e^{-\frac{i}{\hbar}\int_0^T V_I(t')\,dt'}|\psi_0\rangle\right|^2 \end{aligned} \tag{6}$$

where the last equation is written in the interaction picture, and time-ordering is denoted by $\mathcal{T}$. Now we have all the ingredients needed to make the following general, formal claim: in order to be able to turn on and off an interaction strong enough to flip the sign of the wavefunction in (6) over the time T, and to do this accurately enough, so that $p < \epsilon$ (where ϵ is some acceptable error) the state $|\psi_0\rangle$ must have a minimum energy of the order of

$$\langle\psi_0|H_0|\psi_0\rangle_{min} \sim \frac{\hbar}{\epsilon T} \tag{7}$$

This claim involves only the (arbitrary) control system, its self-Hamiltonian, and the interaction V. Note that Eq. (7) is just the same as Eq. (2). The alternate result, Eq. (1), for an oscillator follows from (7) on the assumption that, essentially, the basic "signal to noise" ratio is achieved in one period of oscillation, after which both the "signal" and the "noise" increase together at the same rate (see later examples in this paper, especially Section 3.2.2).

2.2. A counterexample to show that the condition (3) is necessary

It may be useful to show explicitly that the alleged constraint disappears if one allows the interaction to be always "on," that is, if (3) does not hold. Let the control degree of freedom be a harmonic oscillator, let $|\psi_0\rangle = |n\rangle$, an energy eigenstate, and let $V = \hbar g a^\dagger a$. Then one only has to choose $T = \pi/gn$ and equation (6) will be satisfied exactly, with $p = 0$, which means one could make ϵ arbitrarily small, and (7) would be violated.

"Turning the interaction on and off," then, is an essential role played by what we are calling the "control" system here; in fact, it may be used as the distinctive property to identify such a system in any proposed quantum logic scheme. Once identified, the control system must be quantized. A constraint like (1) or (2) will typically not be found to hold if the switching field is treated classically, since such classical fields are normally treated as external parameters, not dynamical variables, and hence are not subject to any back reaction, entanglement, or conservation laws.

Speaking figuratively, the "trick" that the control system (or degree of freedom) has to pull off is to avoid getting entangled with the qubits while turning on and off an interaction that affects different qubit states in different ways. The reason we may expect that this cannot be done with complete success, in general, is that it requires the control system to exert a sort of state-dependent "force" on the qubits, which then react back on it in a way that must also be state dependent. As a result, the two alternative paths in (5) become somewhat distinguishable (by looking at the state of the control) and the coherence of the state of the qubits alone is partly lost.

There is a whole theory (see Ref. 9, Ch. 19, and references therein) of "quantum nondemolition" measurements which are based on "back-reaction avoiding" interactions (although in this case the term "back reaction" is used in the opposite way, to denote the reaction of the meter on the system being measured). The example above, in which the initial state of the "meter" (the "control," in our terminology) is an eigenstate of the interaction Hamiltonian, is actually of this form, and shows why, in general, such schemes will not be applicable here: namely, in order to be able to turn the interaction on and off, the state of the control system *cannot* be an eigenstate of the interaction Hamiltonian. This point will be further elaborated in Section 2.4 below.

2.3. A heuristic proof of (7) when the control is a material particle

Suppose that the "control" system is actually the center of mass coordinate of a material particle (which could itself be one of the physical qubits; the "quantum bits" would then be stored in other degrees of freedom, and the gate could be the result of a collision, which is arranged in such a way that if the two qubits are in the state $|00\rangle$, there is no phase change to the total wavefunction, whereas there is a phase change of π if they are in the state $|11\rangle$; schemes of this type have been proposed, for instance, in Ref. 10). For such systems, an easy way to "derive," heuristically, the constraint (7), along the lines just discussed, is as follows.

If the initial state of the qubits is $|11\rangle$, the potential V produces a force dV/dx on the particle, which, acting over a time T, results in a position change (relative to the unperturbed wavepacket) of

$$\delta x \sim \frac{1}{2m}\frac{dV}{dx}T^2 \tag{8}$$

and a momentum change

$$\delta p \sim \frac{dV}{dx} T \tag{9}$$

From the condition

$$\int_0^T \langle V_I(t') \rangle dt' = \pi\hbar \tag{10}$$

which would have to hold, at least approximately, in order for the desired phase change to take place in (6) (the expectation value is taken in the state $|\psi_0\rangle$; recall we are working in the interaction picture), one can estimate V as $\sim \pi\hbar/T$ and dV/dx as $\sim \pi\hbar/LT$, where L is a characteristic length, that the particle traverses in the time T (so the velocity $v \sim L/T$). Note that $dV/dx \sim \pi\hbar/LT$ directly follows from the requirement that the interaction must be "turned on and off" over the distance L. The center of mass motion of the otherwise freely moving particle acts as the "control handle" that turns the interaction on and off simply by crossing that region of space where V is nonzero.

This position and momentum change of the wavepacket associated with the state $|11\rangle$ will lead to a "mis-overlap" with the wavepacket associated with the state $|00\rangle$, of the order of $(\delta x/\Delta x)^2$ and $(\delta p/\Delta p)^2$, where Δx and Δp are the original, intrinsic position and momentum uncertainty. Then, the requirement that this mis-overlap be negligible (or equivalently "undetectable" to the accuracy specified by ϵ) leads to the constraint

$$\left(\frac{\delta x}{\Delta x}\right)^2 + \left(\frac{\delta p}{\Delta p}\right)^2 < \epsilon \tag{11}$$

which then becomes

$$\left(\frac{\pi\hbar}{L}\right)^2 \left(\frac{T^2}{4m^2}\frac{1}{\Delta^2 x} + \frac{1}{\Delta^2 p}\right) < \epsilon \tag{12}$$

Using the fact that $\Delta x \Delta p \geq \hbar/2$ to optimize (minimize) the left-hand side of (12), we find that it reduces to

$$\frac{2\pi^2\hbar T}{mL^2} < \epsilon \tag{13}$$

which is to say

$$\frac{1}{2}mv^2 > \frac{\pi^2\hbar}{\epsilon T} \tag{14}$$

if $v \sim L/T$.

This is clearly consistent with (7), if the Hamiltonian H_0 is simply that of a free particle (in one dimension). More sophisticated examples will be presented, in greater detail, in the following Section, but hopefully this should be enough to provide a plausibility argument for why one might expect (7) to hold in general.

2.4. Some further insights on the reason for the constraint (7)

The previous subsections attribute the constraint (7) to the state-dependent back-reaction of the qubits onto the control degree of freedom. In a way, this is a very old idea, as old as quantum mechanics: in order to be able to observe interference (i.e., to preserve coherence) in a quantum mechanical system that is interacting with a classical "apparatus," the apparatus (in this case, the "control" system described by $|\psi_0\rangle$) must be large enough for the "back reaction" of the quantum system on it to be negligible.

A somewhat more formal way to argue along these lines might be as follows. Putting together (3) and (6), one can say that the action of the self-Hamiltonian H_0 on $|\psi_0\rangle$ must change it, in a time of the order of T, from a state for which $V|\psi_0\rangle \simeq 0$ to a state for which $V|\psi_0\rangle$ is of the order of $(\pi\hbar/T)|\psi_0\rangle$ (this is a measure of the degree of noncommutativity between H_0 and V). One may, then, expect the V in the exponent of (6) to have a similar effect, and hence to "displace" the energies of the states making up $|\psi_0\rangle$ by an approximate amount $\Delta E \sim \pi\hbar/T$. The condition (7), then, would express the minimum energy that $|\psi_0\rangle$ must have in order to still overlap with itself to the degree given by ϵ, after its component energies have been "messed up" by an amount of the order of ΔE; in this language, it simply reads $\Delta E/E \leq \epsilon$. A simple example of how this may work in practice was given in the previous subsection.

On the other hand, an attempt to evaluate (6) perturbatively might suggest a somewhat different interpretation. To begin with Eq. (3), clearly, if $V_I(t)|\psi_0\rangle$ was nearly equal to zero at all times the interaction would have essentially no effect; so then $|\psi_0\rangle$ cannot, in general, be an eigenstate of $V_I(t)$, which means that $V_I(t)$ will not be sharply defined at all times in the state $|\psi_0\rangle$. There will be fluctuations, which one could formally separate out as

$$V_I(t) = \langle V_I(t)\rangle + \Delta V_I(t) \tag{15}$$

with $\langle V_I(t)\rangle \equiv \langle\psi_0|V_I(t)|\psi_0\rangle$, and $|\psi_0\rangle$ not an eigenstate of ΔV. In that case, if one chooses V, $|\psi_0\rangle$, and T so that

$$\int_0^T \langle V_I(t')\rangle dt' = \pi\hbar \tag{16}$$

one may estimate the failure probability p given by (6), by expanding the exponential, as

$$p \simeq \frac{1}{2\hbar^2}\int_0^T dt \int_0^t dt' \langle\psi_0|\Delta V_I(t)\Delta V_I(t')|\psi_0\rangle + c.c. \tag{17}$$

This is the approach to (approximately) evaluating (6) that was adopted in Ref. 7, and it will also be used in most of the examples to follow.

Equation (17) might be interpreted as suggesting that the reason the gate does not completely work is (at least to this order in perturbation theory) entirely due to the quantum fluctuations in the control system, which cause $V_I(t)$ to be not completely defined as a c-number (i.e., it is an operator, with nonvanishing variance in the state $|\psi_0\rangle$). There is no question, I think, that these intrinsic fluctuations are a significant part of the story, but one should note that the operators $\Delta V_I(t)$ appear in (17) evaluated at different times, so there is more to it than just the variance of $V_I(t)$ at any given time. In fact, if the expression (15) is substituted in (5) (or its interaction-picture equivalent), one can clearly see that both the entanglement and the back reaction discussed earlier are present there, even at this level of perturbation theory.

The examples worked out in the next Section show that, in general, Eq. (17) evaluates to something proportional to the variance of a generalized coordinate in the state $|psi_0\rangle$, times the square of a time integral of some function or derivative of $\langle V\rangle$. This shows why a simple-minded estimate based on just a consideration of the intrinsic fluctuations of the interaction V in the state $|\psi_0\rangle$ typically yields a correct order of magnitude. It also shows (see especially Section 3.2.2) how something like Eq. (1) comes about for a system of oscillators.

3. EXAMPLES

3.1. Switching by a (multimode) electromagnetic field

The first inquiries[1–3] on the effects of the quantum nature of the control system focused on gates driven by electromagnetic fields. Of these works, Ref. 1 dealt only with the entanglement between the field and the qubit (an atom), whereas Ref. 3, while acknowledging the entanglement problem, derived error probability estimates entirely from considerations of the intrinsic fluctuations in the amplitude and (especially) the phase of the field. Additionally, the estimates in Ref. 1 and Ref. 3 are carried out in the context of a "single effective mode" approach. The work by Barnes and Warren,[2] by contrast, involves a complete multimode calculation, in order to describe the temporal evolution of the field adequately. While this makes the problem, formally, much more involved, it yields essentially the same order-of-magnitude estimates for the error probability as the simpler analyses of Ref. 1 and Ref. 3, all of which are in agreement with Eq. (1). Inasmuch as all these papers dealt explicitly only with gates which, like the CNOT or Hadamard gates, typically do not conserve energy and/or angular momentum (unless they are done on encoded qubits, as mentioned in the Introduction), this result is, in fact, only to be expected, according to Ozawa's theorem[4]: the number of "ancillary bosonic qubits" needed to perform the gate should go as $1/\epsilon$, and, in the case of the electromagnetic field, each one of those bosons (a photon) will have an energy of the order of $\hbar\omega$, where ω is some central "carrier frequency."

For this reason, in Ref. 7, I chose to look at a totally hypothetical model in which a coupling of the form $H_I = V|11\rangle\langle 11|$, introduced in subsection 2.1, is mediated by an electromagnetic field, with $V_I(t)$ (in the interaction picture) being some kind of simple function of the (multimode) electric field operator. Although this does not describe exactly any scheme that I know of, it lends itself to simple analytical treatment and illustrates the general consequences of the requirement $p < \epsilon$, when taken together with (3). Moreover, the case of a quadratic coupling is probably relevant to schemes involving the Kerr effect to achieve conditional phase gates.

3.1.1. Linear coupling

Assume linear coupling, and a multimode coherent state:

$$H_I = V_I(t)|11\rangle\langle 11| \equiv \hbar\left(\sum_k g_k a_k e^{-i\omega_k t} + H.c\right)|11\rangle\langle 11| \tag{18}$$

$$|\psi_0\rangle = \prod_k |\alpha_k\rangle. \tag{19}$$

The condition (16) then reads

$$\int \sum_k g_k \alpha_k e^{-i\omega_k t} dt + c.c. = \pi \tag{20}$$

When (17) is used to evaluate the error probability p, and $p < \epsilon$ is required, one obtains the condition

$$\sum_k \left|\int g_k e^{-i\omega_k t} dt\right|^2 < \epsilon \tag{21}$$

However, from (20) we get that

$$\sum_k |\alpha_k| \left|\int g_k e^{-i\omega_k t} dt\right| \geq \frac{\pi}{2} \tag{22}$$

Then (21) is only possible if

$$\left(\sum_k |\alpha_k|^2\right)^{1/2} \geq \frac{\pi}{2\sqrt{\epsilon}} \tag{23}$$

The pulse's "average frequency" is

$$\langle\omega\rangle = \frac{\sum_k \omega_k |\alpha_k|^2}{\sum_k |\alpha_k|^2} \leq \frac{4\epsilon}{\pi^2}\sum_k \omega_k |\alpha_k|^2 \tag{24}$$

and the total pulse energy is

$$E_{field} = \sum_k \hbar\omega_k |\alpha_k|^2 \tag{25}$$

so

$$E_{field} \geq \frac{\pi^2}{4}\frac{\hbar\langle\omega\rangle}{\epsilon} \tag{26}$$

For an oscillatory field, with a well-defined "carrier frequency" ω_0, this is essentially Eq. (1). I believe that Eq. (26) should also hold for a "static" field, switched on and off over a time T, in which case $\langle\omega\rangle \sim 1/T$, and we get Eq. (2).

3.1.2. Nonlinear coupling

More generally, assume that the Hamiltonian is of the form:

$$H_I = \hbar g \mathcal{E}^q(t)|11\rangle\langle 11| \tag{27}$$

with g a time-independent coupling constant, and q an arbitrary integer. Let $\mathcal{E}(t)$ be the multimode electric field operator

$$\mathcal{E}(t) = \sum_k \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} a_k e^{-i\omega_k t} + H.c \tag{28}$$

It is understood that the sum over frequencies is limited by the natural frequency response of the system. Let $\mathcal{E} = \langle\mathcal{E}\rangle + \Delta\mathcal{E}$. Then we can write

$$\begin{aligned}\Delta V_I \quad &= \hbar g q \langle\mathcal{E}\rangle^{q-1}\Delta\mathcal{E} \\ &= \hbar g q \langle\mathcal{E}\rangle^{q-1} \\ &\quad \times \sum_k \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \left(\Delta a_k e^{-i\omega_k t} + \Delta a_k^\dagger e^{i\omega_k t}\right)\end{aligned} \tag{29}$$

The error in the operation of the gate can be estimated, as before, using (17):

$$\frac{1}{2\hbar^2}\int_0^T dt \int_0^t dt' \langle\psi_0|\Delta V_I(t)\Delta V_I(t')|\psi_0\rangle + c.c. = \sum_k \left| qg\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \int \langle\mathcal{E}\rangle^{p-1} e^{-i\omega_k t} dt\right|^2 \tag{30}$$

whereas, on the other hand, we want

$$g\int \langle\mathcal{E}\rangle^q dt = \pi \tag{31}$$

and the left-hand side of this expression can be written as

$$\sum_k g\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \int \langle\mathcal{E}\rangle^{q-1}\alpha_k e^{-i\omega_k t} dt + c.c. \tag{32}$$

The two conditions

$$\sum_k \alpha_k \int g\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \langle\mathcal{E}\rangle^{q-1} e^{-i\omega_k t} dt + c.c. = \pi \tag{33}$$

and

$$\sum_k \left|\int g\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \langle\mathcal{E}\rangle^{q-1} e^{-i\omega_k t} dt\right|^2 < \frac{\epsilon}{q^2} \tag{34}$$

are formally equivalent to (20) and (21) with a time-dependent g_k and a modified ϵ, and so the same logic applies to yield for the total field energy

$$E_{field} \geq \frac{\pi^2}{4}\frac{q^2\hbar\langle\omega\rangle}{\epsilon} \tag{35}$$

As long as $q^2 \geq 1$ (i.e., the Hamiltonian is an analytic function of the field) this condition is at least as restrictive as (26).

3.1.3. Squeezing?

In a coherent state, both "quadratures" of the field-amplitude operator a_k have the same noise. One could imagine a Hamiltonian that couples only to one quadrature, which could then be squeezed.

What might happen then could be roughly as follows. The fluctuations (squared) in the squeezed quadrature would be reduced by a factor e^{-2r}, where r is the squeezing parameter. This could amount to formally increasing

ϵ in Eq. (21) by a factor e^{2r}. Note that the number of photons in the field is now given by $|\alpha_k|^2 + e^{2r}$, so, in fact, the equation for the field energy might end up reading

$$E_{field} \geq \hbar\langle\omega\rangle \left(\frac{1}{e^{2r}\epsilon} + e^{2r}\right) \tag{36}$$

When this expression is minimized over r, one obtains

$$E_{field} \geq \frac{2\hbar\langle\omega\rangle}{\sqrt{\epsilon}} \tag{37}$$

Note, however, that to couple to a squeezed field one typically needs a local oscillator at the carrier frequency ω. Presumably, if ω is not sufficiently sharply defined, errors in the gate operation will result. This means that broadening of ω due to the finite pulse duration must be prevented. If the condition

$$(\omega T)^2 > \frac{1}{\epsilon} \tag{38}$$

is applied to equation (37), one obtains again

$$E_{field} \geq \frac{\hbar}{\epsilon T} \tag{39}$$

Thus, it seems that even using squeezing one is still constrained by the inequality (39). A more careful study of this possibility, however, may be necessary, ideally in the context of a specific model for the coupling interaction.

3.2. Switching using collisions between wavepackets

3.2.1. "Free" particles

Suppose one arranges to have a collision between the two particles involved in the gate operation, with the idea that their mutual interaction, $V(|\mathbf{r}_1 - \mathbf{r}_2|)$, will provide the desired phase shift. Work in the center of mass frame assuming identical particles; neglect deviations of the particles' motion from straight lines at constant speed; let b be the distance of closest approach and take that to be the x direction. Then we can approximate Eq. (16) as

$$\int_0^T \langle V_I(t')\rangle dt' \simeq \int_{-T/2}^{T/2} V(\sqrt{4v^2t^2 + b^2})dt \equiv \int_{-T/2}^{T/2} V(\rho(t))dt = \pi\hbar \tag{40}$$

where $\pm v$ is the y-component of the particles' velocity in the CM frame, and $\rho = (4v^2t^2 + b^2)^{1/2}$.

In practice the free wavepackets' x coordinate is uncertain by an amount equal to

$$\Delta x(t) = \Delta x_0 + \frac{\Delta p_0}{m}\left(t + \frac{T}{2}\right) \tag{41}$$

assuming that x and p are initially uncorrelated (at the time $t = -T/2$). We can use this, in an expansion of V, to approximate ΔV_I as $(dV/d\rho)(d\rho/db)\Delta x$ which, when substituted in Eq. (17), yields, for the error probability,

$$p = \frac{b^2}{\hbar^2}\left(\int \frac{dV}{d\rho}\frac{dt}{\rho}\right)^2 \left(\Delta x_0^2 + \frac{T^2\Delta p_0^2}{4m^2}\right) \tag{42}$$

with $\rho = (4v^2t^2 + b^2)^{1/2}$, and making use of the symmetry of the integrands.

One can now minimize p with the constraint $\Delta x_0 \Delta p_0 \geq \hbar/2$ (i.e., pick an optimal wavepacket), with the result $\Delta x_0^2 = T\hbar/4m$, $\Delta p_0^2 = 2m\hbar/T$. Then the condition $p < \epsilon$ becomes

$$\frac{b^2}{\hbar^2}\frac{T\hbar}{2m}\left(\int \frac{dV}{d\rho}\frac{dt}{\rho}\right)^2 < \epsilon \tag{43}$$

Now use (40) to eliminate the first factor of $1/\hbar^2$, and consider the derivative of the left-hand side of (40) with respect to the impact parameter b. One easily obtains the constraint

$$\frac{\pi^2 T\hbar}{2m}\left[\frac{d}{db}\ln\left(\int_{-T/2}^{T/2} V(\sqrt{4v^2t^2+b^2})dt\right)\right]^2 < \epsilon \tag{44}$$

This can be simplified a bit by, first, introducing the obvious change of variable $2vt = y$, with limits of integration $\pm y_0 = \pm vT$, and then assuming that y_0 is large enough that no substantial error is introduced by extending the integration to $\pm\infty$. This is reasonable, since the particles have to start and end far enough away from each other for their mutual interaction to be negligible. We then have

$$\frac{\pi^2 T\hbar}{2m}\left[\frac{d}{db}\ln\left(\int_{-\infty}^{\infty} V(\sqrt{y^2+b^2})dy\right)\right]^2 < \epsilon \tag{45}$$

The left-hand side of (45) is easily evaluated when $V(\rho)$ is any power law, since one can write $(y^2+b^2)^{-n/2} = b^{-n}((y/b)^2+1)^{-n/2}$ and then the change of variable $y/b = u$ results in a factor of b^{-n+1} times an integral which is independent of b. Hence for any $n > 1$, we get

$$\frac{\pi^2 T\hbar}{2m}\frac{(n-1)^2}{b^2} < \epsilon \tag{46}$$

and, since, as argued above, we must have $b < y_0 = vT$, this yields immediately

$$\frac{\hbar}{mv^2T} < \epsilon \tag{47}$$

or

$$mv^2 > \frac{\hbar}{\epsilon T} \tag{48}$$

where mv^2 is the initial kinetic energy of the two particles.

3.2.2. Particles in a harmonic potential

To prevent spreading of the wavepackets during the interaction, one could imagine confining the particles in a static potential (a time-dependent potential implies a time-dependent field, and we are back to the previous Section). Assume the potential is harmonic, and consider the following scenario: at time $t = 0$ we create the two wavepackets, a distance $4A + b$ apart, let them oscillate towards each other with amplitudes A, so that, at the time $t = \pi/\omega$, they are closest, a distance b apart; then they swing back to the starting position by the time $t = 2\pi/\omega$. At a minimum, one needs to put in and remove enough energy to start and stop this pendulum motion. Just how this is done is left vague for the moment, but it might be important later on.

In any case, assume that we have an interaction energy $V(\rho)$, as before, but now only in one dimension, with

$$\begin{aligned} x_1 &= -\left(A+\frac{b}{2}\right) - A\cos\omega t \\ x_2 &= \left(A+\frac{b}{2}\right) + A\cos\omega t \end{aligned} \tag{49}$$

and $\rho = x_2 - x_1 = 2A + b + 2A\cos\omega t$. The desired action is again, approximately,

$$\frac{1}{\hbar}\int_0^{2\pi/\omega} V(\rho)dt = \pi \tag{50}$$

The error operator Δx for a harmonic oscillator is

$$\Delta x = \Delta x_0 \cos\omega t + \frac{\Delta p_0}{m\omega}\sin\omega t \tag{51}$$

so we find, proceeding as before,

$$
\begin{aligned}
p &= \frac{1}{\hbar^2}\left[\left(\int \frac{dV}{d\rho}\cos\omega t\,dt\right)^2 \Delta^2 x_0 + \left(\int \frac{dV}{d\rho}\sin\omega t\,dt\right)^2 \frac{\Delta^2 p_0}{m^2\omega^2}\right] \\
&= \frac{1}{\hbar^2}\left(\int \frac{dV}{d\rho}\cos\omega t\,dt\right)^2 \Delta^2 x_0
\end{aligned}
\tag{52}
$$

The contribution of Δp_0 vanishes due to the symmetry of the integration, which raises again the possibility of using squeezing to improve on the constraint to be derived presently. For the moment, however, assume simply that a coherent state wavepacket is excited; then Δx_0 is just the ordinary zero-point fluctuation of the ground state of a harmonic oscillator, $\Delta^2 x_0 = \hbar/2m\omega$, and now the constraint we have is

$$
\frac{\pi^2\hbar}{2m\omega}\frac{\left(\int_0^{2\pi/\omega} \frac{dV}{d\rho}\cos\omega t\,dt\right)^2}{\left(\int_0^{2\pi/\omega} V(\rho)\,dt\right)^2} < \epsilon \tag{53}
$$

The integrals in this case do not seem so easy to evaluate in general, but specific cases can readily be done. For instance, for a dipole-dipole ρ^{-3} interaction one finds to leading order in b

$$
\frac{\pi^2\hbar}{2m\omega}\left(\frac{5}{2b}\right)^2 < \epsilon \tag{54}
$$

Note that the energy of each oscillator is $\frac{1}{2}m\omega^2 A^2$, that the interaction time $T = 2\pi/\omega$, and that, as before, we'll want $A > b$, so again we find that

$$
\frac{1}{2}m\omega^2 A^2 > \frac{50\pi^3}{16}\,\frac{\hbar}{\epsilon T}. \tag{55}
$$

Note that we could, equivalently, envision a situation in which the same effect is achieved over a large number (say, N) of cycles, so that $V(\rho)$ in (50) is reduced by a factor of N and the integral extends to a time $2\pi N/\omega$. Because of the periodicity of the motion, however, all such integrals (particularly in (53)) are equal to just N times the integral over a single period; all the factors of N cancel from (53), and one is left again with the result (54), which now leads to

$$
\frac{1}{2}m\omega^2 A^2 > \frac{25\pi^2}{16}\,\frac{\hbar\omega}{\epsilon} \tag{56}
$$

This shows explicitly how the constraint (1) originates. Going over the derivation, one can see that ω appears in it because it characterizes the size of the minimum-uncertainty oscillator wavepacket (coherent state), and provides a connection between this quantity and the oscillator's energy.

As mentioned above, it looks as if one could use a squeezed state (squeezed in the position variable) to improve on the constraint (55). This is because $\Delta^2 p_0$ does not appear in Eq. (52), which in turn follows from the symmetry of the integral over the *unperturbed* trajectory. Numerical calculations done for a classical particle, however, show that (as is only to be expected), as a result of the interaction, the particle does not return exactly to the starting point. The importance of this mismatch between the perturbed and unperturbed wavepackets would only be magnified if the quantum wavepacket was squeezed in position. Hence, there has to be a limitation to how much one can squeeze the position, but it does not seem a simple matter to derive it. Specifically, it seems that, in the formalism used here, these effects would appear to a higher order in the expansion (52) (or, more precisely, in the expansion of Eq. (6) that yielded Eq. (17)).

There may actually be good self-consistency reasons to go to higher orders in Δp^2 or Δx^2, in the case of large squeezing. If one has a state that is squeezed in position, say, enough to change the dependence of the minimum energy on ϵ, from ϵ^{-1} to $\epsilon^{-1/2}$ (the best achievable in any case, by the arguments of Section 3.1.3), the squeezing factor e^{-2r} in $\Delta^2 x_0$ would have to be of the order of $\sqrt{\epsilon}$. In that case, the corresponding factor e^{2r} in $\Delta^2 p_0$ would be of the order of $1/\sqrt{\epsilon}$, and in $\Delta^4 p_0$ it would be of the order of $1/\epsilon$. This suggests that in case of such extreme squeezing, one would not be justified to neglect the higher order terms (in particular, terms of order $\Delta^4 p_0$) in (52). This point probably deserves a closer look.

4. ATOM-LIKE QUBITS AND SPONTANEOUS EMISSION

A common feature of all the studies mentioned at the beginning of Section 3.1 is that spontaneous emission into any modes other than the driving field mode was neglected. More precisely, the atom-like qubit (which, depending on the specific realization, could be an actual atom, an ion, an exciton in a quantum dot, etc.) was taken to interact only with *occupied* modes of the quantized electromagnetic field. This is natural, since the original goal of this research was to find out what decoherence is induced by the quantum nature *of the control field*, and not by anything else. Nonetheless, something very interesting happens when the unoccupied (vacuum) modes are brought into consideration.

It has been known for a long time[11, 12] that a multimode quantum field in a coherent state can be reduced, by means of a unitary transformation, to a classical field plus a set of modes in the vacuum state. In this picture, the decoherence discussed in Section 3.2 (and in Refs. 1–3) would seem to be directly attributable to the "vacuum" modes that replace the modes actually occupied by the control e.m. field. But if this is the case, then one may expect that the other vacuum modes, neglected in the above analysis (because they truly are in the vacuum state) may have a similar decoherence effect: similar, that is, in scaling and in order of magnitude, and different, mostly, by a geometrical factor that takes into account their number, relative to the number of modes occupied by the control field.

Consider, for definiteness, a laser beam of cross-sectional area A interacting with an atom. To describe such a beam as a superposition of transverse plane waves may require a spread in $k_\perp$ of the order of $\Delta k_\perp \sim 2\pi/\sqrt{A}$. In k space, the modes required, therefore, fill a solid angle of the order of $\Delta\Omega = \Delta k_\perp^2/k^2 = \lambda^2/A$. The ratio of the number of such modes to the total number of modes in all directions of space is $\Delta\Omega/4\pi$. Accordingly, it may be expected, on the basis of the picture suggested above, that the decoherence calculated in Section 3.2 is only a fraction, of the order of $\lambda^2/4\pi A$, of the total decoherence due to *all* the quantum modes—both those initially occupied by the control beam, and those initially empty. If this is the case, then to include all the modes one would merely rewrite the error probability, which by (1) is given by $1/\bar{n}$ ($\bar{n}$ being the average number of photons in the pulse), as $4\pi A/\lambda^2\bar{n}$, and the requirement (1) would be replaced by

$$E_{min} \geq \frac{4\pi A}{\lambda^2}\frac{\hbar\omega}{\epsilon} \tag{57}$$

which can be rewritten as a constraint on the minimum energy density (energy per unit area) in the beam:

$$\left(\frac{E}{A}\right)_{min} = \frac{4\pi\hbar\omega}{\epsilon\lambda^2}. \tag{58}$$

The quantity on the left-hand side of (58) is a useful one to work with, because it is directly proportional to the magnitude of the classical Poynting vector, and hence to the square of the classical electric field.

Here is, however, yet another alternative way to look at things. The effect of the modes which are truly in the vacuum state originally can only be to provide an avenue for spontaneous emission to take place. Hence, if the reasoning above is correct, it follows that one should be able to get a constraint of the form (58) merely from the requirement that spontaneous emission be negligible during the time it takes to perform the logical gate.

This is indeed the case, as can be seen from the following. Let the spontaneous emission rate be Γ, and let the time needed to perform a logical operation on the atomic qubit be T. Then, if ϵ is the tolerable error probability, we clearly require

$$\Gamma T < \epsilon \tag{59}$$

which means that one needs to apply a sufficiently strong field to the atom (we are reasoning semiclassically now), since T will be inversely proportional to the Rabi frequency Ω_R. Let's assume, for the sake of argument, that $T = \pi/\Omega_R$. Then (59) can be rewritten as

$$\frac{\pi^2 \Gamma}{\Omega_R^2 T} < \epsilon \tag{60}$$

or, using standard expressions[9] for Γ and Ω_R,

$$\frac{\pi^2}{4\pi\epsilon_0} \frac{4\omega^3 d_{ab}^2}{3\hbar c^3} \left(\frac{d_{ab}\mathcal{E}_0}{\hbar}\right)^{-2} < \epsilon T \tag{61}$$

(where d_{ab} is the atomic dipole moment and $\mathcal{E}_0$ the field amplitude). Now use $\omega = 2\pi c/\lambda$ and note that the e.m. field's energy density is $\frac{1}{2}\epsilon_0\mathcal{E}_0^2$. One gets the constraint

$$\frac{1}{2}\epsilon_0\mathcal{E}_0^2 \left(\sigma_{tot}\, cT\right) > \pi^2 \, \frac{\hbar\omega}{\epsilon} \tag{62}$$

where $\sigma_{tot} = 3\lambda^2/2\pi$ is the total scattering cross-section for an atom in free space, and in resonance (see Ref. 12, p. 533). In words, the number of photons within a volume V of cross-sectional area σ_{tot} and length cT (the length of the pulse) has to be greater than $1/\epsilon$. Note that this condition actually implies Eq. (1), since, due to diffraction, it would not realistic to expect that the pulse could be focused to a cross-sectional area smaller than σ_{tot}; hence, the total energy in the whole pulse must be greater than the right-hand side of (62).

The required energy per unit volume, $\frac{1}{2}\epsilon_0\mathcal{E}_0^2$, follows from (62) by dividing both sides by $\sigma_{tot}\, cT$, and the energy per unit area is obtained then by multiplying both sides by cT, the length of the pulse. The result is

$$\left(\frac{E}{A}\right)_{min} = \frac{2\pi^3\hbar\omega}{3\epsilon\lambda^2}. \tag{63}$$

which is the same order of magnitude as Eq. (58). Hence, we can obtain Eq. (58) in two ways: either by the conventional requirement that spontaneous emission be negligible during the performance of the quanum logical gate, or by applying to the right-hand side of Eq. (1) a correction factor of the order of $\lambda^2/4\pi A$, which is meant to represent the fraction of modes of the quantized e.m. field occupied by the "control" beam.

There are a couple of ways to look at this result. One is to say that the quantum nature of the control field does not, in fact, introduce any constraints on the performance of the gate, beyond those that were already implicit in the requirement that spontaneous emission *in all available modes* during the gate operation should be negligible. This is true, but it could mislead one into thinking that those constraints can be made negligible by setting up a situation (such as an atom in a microcavity) in which spontaneous emission into the originally unoccupied modes is strongly suppressed. As this is not the case, it is probably better, and less misleading, to say that the constraint (1) correctly accounts for that part of the total decoherence due to the quantum nature of the control beam, *which must be there in any case*, even if spontaneous emission in all the *other* modes is suppressed somehow.

Finally, and for completeness, it may be useful to point out that Eq. (62) also holds for the case, treated in Ref. 3, of a transition driven by detuned Raman lasers. In that case, it can be argued that the atom only spends a time of the order of $1/\Delta$ in the excited (intermediate) state, and hence the loss of purity due to spontaneous emission would be limited to Γ/Δ, regardless of T; but, for large detuning, one has also an effective two-photon Rabi frequency of the order of Ω_R^2/Δ, so by combining $\Gamma/\Delta < \epsilon$ with $\Omega_R^2 T/\Delta = \pi$, and eliminating Δ, one immediately obtains Eq. (60), and the rest proceeds as above.

5. DISCUSSION AND CONCLUSIONS

The possible implications of the results presented here for hypothetical very large-scale quantum computations have been already discussed at some length in Ref. 7. Here only a couple of order-of-magnitude estimates will be given.

Suppose one uses as the control system an electromagnetic field at optical frequencies ($\omega \sim 10^{15}$ rad/s). Let the tolerable failure probability $\epsilon \sim 10^{-5}$. Then the minimum energy per logical operation required by (1) is of the order of 10^{-14} J. The corresponding energy density (63) would be of the order of 1 J/m^2. To convert this to power, one would need to know the duration of the gate, T, which in turn would be constrained by the decoherence time τ_c: one must have $T < \epsilon\tau_c$. For a coherence time of the order of 1 ms, therefore, one would have $T < 10^{-8}$ s and hence a minimum intensity (power per unit area) of the order of 10^8 W/m^2, or 10^4 W/cm^2. In empty space may not be excessive, but in a solid-state system (which is likely to have an even shorter decoherence time) it may well be.

Generally speaking, the considerations presented here favor systems that have long decoherence times and rely on static, rather than oscillatory, fields for the performance of the logical gates.

In any case, one should note that the inverse relationship (2) between the speed of a gate and the energy needed to carry it out indicates that quantum computers will not exhibit the same trend of decreasing size and increasing speed which conventional computers have exhibited under Moore's law.

There are still open questions regarding the theoretical results presented here: how difficult would a general, formal, rigorous proof be? Is the main claim (7) provable only for systems, like harmonic oscillators and free particles, where the energy is a quadratic function of generalized coordinates? If so, is it only true for coherent states, or could "nonclassical" states, such as squeezed states, be more useful? Interesting as these questions are, from a theoretical point of view, their practical relevance is somewhat limited, since the present results already appear to cover all important systems, and the generation of nonclassical states is typically a very inefficient process, so it is unlikely to lead, in practice, to a reduction in the energy requirements.

ACKNOWLEDGMENTS

I am grateful to Wayne Itano for pointing out Refs. 11, 12, and suggesting that, in view of those results, the total decoherence due to spontaneous emission would probably always have to be larger than the one due to the quantum nature of the control field alone.

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