Historical Comments: Historically, the study of differential equations originated with the introduction of the calculus by Sir Isaac Newton (1642-1727) and Gottfried Wilhelm von Leibniz (1646-1716). Although mathematics began as a recreation for Newton, he became known as a great mathematician by the age of 24 after his invention of the calculus, discovery of the law of universal gravitation, and experimental proof that white light is composed of all colors. Leibniz completed his doctorate in philosophy by the age of 20 at the University of Altdorf. Afterward he studied mathematics under the supervision of Christian Huygens (1629-1695) and, independently of Newton, helped develop the calculus. Leibniz corresponded regularly with other mathematicians concerning differential equations, and he developed several methods for solving first-order equations.

Other prominent mathematicians who contributed to the development of differential equations and their applications were members of the famous Bernoulli family of Switzerland, the most famous of which are James (1654-1705) and John (1667-1748). Over the years there have been a host of mathematicians who contributed to the general development of differential equations.

The objective of this chapter is to review the basic ideas found in a first course in ordinary differential equations (ODEs). In doing so, we will concentrate primarily on those we deem most important in engineering applications. Because DEs are considered the most fundamental models that are used in a wide variety of physical phenomena, they play a central role in many of the following chapters of this text.

1.1 Introduction

Differential equations (DEs) play a fundamental role in engineering and science because they can be used in the formulation of many physical laws and relations. The development of the theory of DEs is closely interlaced with the development of mathematics in general, and it is indeed difficult to separate the two. In fact, most of the famous mathematicians from the time of Newton and Leibniz had a part in the cultivation of this fascinating subject. The first problems studied that involved the notion of DE came from the field of mechanics. Consequently, some of the terminology that persists today (like "forcing function") had its beginning in these early mechanics problems.

At its most basic level, *Newton's second law of motion* is commonly expressed by the simple algebraic formulation

$$F = ma.$$

For a "particle" or body in motion, F denotes the force acting on the body, m is the mass of the body (generally assumed to be constant), and a is its *acceleration*. In practice, however, it is the *velocity* and *position* of the body as a function of time that may be more useful. Recalling that the velocity (speed) v is related to acceleration by $a = dv/dt$, Newton's second law may also be expressed by

$$F = m\frac{dv}{dt}.$$

Last, the position y of the body is related to velocity by $v = dy/dt$, and by substituting this last expression into the above equation, we get another variation of Newton's law given by

$$F = m\frac{d^2y}{dt^2}.$$

These last two expressions of Newton's second law are considered DEs because they involve derivatives of unknown functions. And although the study of DEs grew out of certain kinds of problems in mechanics, their use today is far more widespread. For example, they occur in various branches of engineering and physical science to study problems in the following areas (among others):

- the study of particle motion
- the analysis of electric circuits and servomechanisms
- continuum and quantum mechanics
- the theory of diffusion processes and heat flow
- electromagnetic theory

Other disciplines such as economics and the biological sciences are also using DEs to investigate problems like the following:

- interest rates
- population growth
- the ecological balance of systems
- the spread of epidemics

1.2 Classifications

Definition 1.1 By *differential equation* we mean an equation that is composed of

- a single unknown function y, and
- a finite number of its derivatives.

A simple example of a DE found in the calculus is to find all functions y for which

$$y' = f(x), \tag{1}$$

where $y' = dy/dx$ and $f(x)$ is a given function. The formal solution to this equation is

$$y = \int f(x)dx + C, \tag{2}$$

where C is an arbitrary constant.

Most of the DEs that concern us are not of the simple type illustrated by (1). Typical examples of the types of equations found here include

$$y' = x^2 y^3 \tag{3}$$

$$y'' + cy' + k^2 y = \sin x \tag{4}$$

$$y'' + b\sin y = 0 \tag{5}$$

$$(y')^2 + 3xy = 1 \tag{6}$$

In order to provide a framework in which to discuss various solution techniques for DEs, it is helpful to first introduce *classification schemes* for the equations. For example, some important classifications are the following:

- **Order:** The *order* of a DE is the order of its highest derivative.
- **Linear:** The DE is said to be *linear* if it is linear in the unknown function y and all derivatives. If a DE is not linear, it is called *nonlinear*.
- **Ordinary DE:** A DE is called *ordinary* when the unknown function depends on only one independent variable. Otherwise, it is a *partial* DE.

Based on these definitions, Eqs. (3)-(6) are *ordinary*, Eqs. (3) and (6) are *first-order* equations, and (4) and (5) are *second-order*. Also, only Eq. (4) is *linear*—the others are *nonlinear*.

1.2.1 Solutions of differential equations

Definition 1.2 A *solution* $y = y(x)$ of a DE on an interval I is a continuous function possessing all derivatives occurring in the equation that, when substituted into the DE, reduces it to an identity for all x in the interval I.

EXAMPLE 1 Verify that $y = e^{-x}$ is a *solution* of the first-order DE

$$y' + y = 0.$$

Solution: To verify that a given (differentiable) function is a solution of a DE, we simply substitute it directly into the DE. Note that the function $y = e^{-x}$ is continuous and has continuous derivatives for all x. Furthermore,

$$y' + y = -e^{-x} + e^{-x} = 0$$

for all values of x. It also follows in the same manner that $y = C_1 e^{-x}$ is a solution for all values of x and all values of the constant C_1.

EXAMPLE 2 Verify that $y_1 = C_1 \cos 2x$ and $y_2 = C_2 \sin 2x$ are both solutions of the second-order DE

$$y'' + 4y = 0$$

for any values of the constants C_1 and C_2.

Solution: Both functions are continuous and have continuous derivatives for all x. For y_1, we have

$$y_1'' + 4y_1 = -4C_1 \cos 2x + 4C_1 \cos 2x = 0,$$

for all values of x. Similarly, for y_2 it follows that

$$y_2'' + 4y_2 = -4C_1 \sin 2x + 4C_1 \sin 2x = 0.$$

In Example 2, we illustrated that both y_1 and y_2 are solutions of the given DE. Moreover, it is easy to verify that the functions

$$y = C_1 \cos 2x + C_2 \sin 2x,$$

$$y = C_3 \sin x \cos x,$$

are also solutions of the same DE.

Solutions are classified in the following manner:

- **Particular solution:** If a solution contains no arbitrary constants, it is called a *particular solution* of the DE.

- **General solution:** A function that contains all particular solutions of the DE is called a *general solution*.

Because of arbitrary constants, the above DEs (Examples 1 and 2) have infinitely many solutions. However, the *number of arbitrary constants* that appears in a general solution is always *equal to the order of the DE.*

In solving DEs it can be important to know in advance: “Does a solution *exist*?” If so, we may then also want to know: “Is it *unique*?” In general, questions concerning the existence and uniqueness of solutions can be very difficult to answer.

1.3 First-Order Equations

First-order DEs arise naturally in problems involving the determination of the velocity of free-falling bodies subject to a resistive force, finding the current or charge in an electric circuit, finding curves of population growth, and in radioactive decay, among other applications. Each type of first-order DE that arises in practice may demand a different method of solution. And although others exist, we will introduce only two methods of solution—*separation of variables* and *linear equations*—both of which are applicable to a wide variety of practical problems involving first-order DEs.

First-order DEs are typically written in either the *derivative form*

$$y' = F(x,y) \tag{7}$$

or, through formal manipulations, in the *differential form*

$$M(x,y)dx + N(x,y)dy = 0. \tag{8}$$

In applications, the solution of (7) or (8) is usually required to also satisfy an *auxiliary condition* of the form

$$y(x_0) = y_0, \tag{9}$$

which geometrically specifies that the graph of the solution pass through the point (x_0,y_0) of the xy-plane. Because x_0 is often the beginning point in the interval of interest, the condition (9) is also called an *initial condition* (IC). Hence, solving (7) or (8) subject to the auxiliary condition (9) is called an *initial value problem* (IVP).

Many first-order DEs that routinely arise in applications are *nonlinear*. In some cases these may be very difficult or impossible to solve by known methods. To ensure that the DE together with its initial condition (9) has a solution, we have the following *existence-uniqueness theorem* which we state without proof (see [17]).

Theorem 1.1 If $F(x,y)$ is a continuous function in a rectangular domain $a < x < b$, $c < y < d$ containing the point (x_0,y_0), then the IVP

$$y' = F(x,y), \quad y(x_0) = y_0$$

has at least one solution in some interval $|x - x_0| < h$, $(h > 0)$ embedded in $a < x < b$. If, in addition, the partial derivative $\partial F/\partial y$ is continuous in that rectangle, then the IVP has a *unique* solution.

Remark: The conditions stated in Theorem 1.1 are only *sufficient conditions*—not *necessary conditions*. That is, if these conditions are not satisfied, the problem may have *no solution*, but in some cases may have *more than one solution* or even a *unique solution*!

1.3.1 Separation of variables

Perhaps the easiest method to apply when it is appropriate is that called *separation of variables*. The first-order DE written in differential form

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{10}$$

is said to be "separable" if it can be rearranged in the form

$$\boxed{f(y)\,dy = g(x)\,dx.} \tag{11}$$

Observe that the left-hand side of (11) is a function of y alone and the right-hand side is a function of x alone. A family of solutions can be obtained by simple integration of each side to yield the form

$$\int f(y)\,dy = \int g(x)\,dx + C, \tag{12}$$

where C is a constant of integration (only one such constant is required because the arbitrary constants from each side of the equation can be combined).

EXAMPLE 3 Solve the DE

$$(1-x)\,dy + y\,dx = 0.$$

Solution: We see that division by y and $(1-x)$ leads to the "separated form"

$$\frac{dy}{y} = -\frac{dx}{1-x}, \quad x \neq 1,\ y \neq 0.$$

Thus, by integrating each side independently, we arrive at

$$\ln|y| = \ln|1-x| + C,$$

which, by writing $C = \ln|C_1|$ and using properties of logarithms, yields

$$y = C_1(1-x).$$

Remark: Recall from calculus that $\int \frac{du}{u} = \ln|u| + C$, where the absolute value is retained except in those cases in which we know $u > 0$.

EXAMPLE 4 Solve the IVP

$$(x^2 + 1)y' + y^2 + 1 = 0, \quad y(0) = 1.$$

Solution: By rearranging terms, we have

$$\frac{dy}{y^2 + 1} = -\frac{dx}{x^2 + 1},$$

which, upon integration, leads to

$$\tan^{-1} y = -\tan^{-1} x + C.$$

If we apply the prescribed initial condition $(x = 0,\ y = 1)$, we see that $C = \pi/4$, and consequently,

$$\tan^{-1} y + \tan^{-1} x = \frac{\pi}{4}.$$

A more convenient form of the solution can be obtained by using the trigonometric identity

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Thus, we find

$$\tan(\tan^{-1} y + \tan^{-1} x) = \tan(\pi/4),$$

or, equivalently,

$$\frac{y + x}{1 - xy} = 1.$$

Now, solving explicitly for y, we get

$$y = \frac{1 - x}{1 + x}, \quad x \neq -1.$$

1.3.2 Linear equations

A *linear first-order DE* is any equation that can be put in the form

$$A_1(x)y' + +A_0(x)y = F(x). \tag{13}$$

The functions $A_0(x)$ and $A_1(x)$ are the *coefficients* of the DE (which do not depend on y) and $F(x)$ is the *input function*, also called the *forcing function*. In practice, the solution of (13) is usually required to satisfy the initial condition (IC)

$$y(x_0) = y_0. \tag{14}$$

For developing the solution of a linear DE it is customary to first put (13) in the *normal* *form*

$$\boxed{y' + a_0(x)y = f(x),} \tag{15}$$

obtained by dividing each term of (13) by $A_1(x)$. Hence, $a_0(x) = A_0(x)/A_1(x)$ and $f(x) = F(x)/A_1(x)$.

To develop a general solution of (15), we first consider the associated *homogeneous equation*

$$\boxed{y' + a_0(x)y = 0,} \tag{16}$$

in which the input function $f(x)$ is identically zero. One feature of a homogeneous equation is that $y = 0$ is always a solution, called the *trivial solution*. However, our interest concerns *nontrivial* (nonzero) *solutions*. We do note that (16) can be formally solved by the method of separation of variables, which leads to

$$\frac{dy}{y} = -a_0(x)dx, \quad y \neq 0.$$

The direct integration of this expression then yields

$$\ln|y| = -\int a_0(x)dx + C, \tag{17}$$

where C is a constant of integration. By solving (17) directly for y, we obtain the family of solutions

$$\boxed{y_H = C_1 y_1(x) = C_1 \exp\left[-\int a_0(x)dx\right],} \tag{18}$$

where we write y_H to denote that (18) is a *solution* of the *homogeneous* equation. We also introduced the notation $C_1 = e^C$, and the function $y_1(x)$ is defined by

$$y_1(x) = \exp\left[-\int a_0(x)dx\right]. \tag{19}$$

The solution function (18) is a *general solution* of the *homogeneous equation* (16). A *general solution* of the *nonhomogeneous equation* (15) is defined by the sum $y = y_H + y_P$, where y_P is any *particular solution* and y_H is defined above. To construct the particular solution, we employ a method called *variation of parameters* that was developed by J. L. Lagrange (1736-1813). Specifically, we look for a solution of the form

$$y_P = u(x)y_1(x), \tag{20}$$

where $u(x)$ is a function to be determined. The technique derives its name from the fact that the arbitrary constant C_1 in the homogeneous solution (18) is replaced by the unknown function $u(x)$.

The direct substitution of (20) into the left-hand side of the DE in (15) gives us

$$\begin{aligned} y_P' + a_0(x)y_P &= \frac{d}{dx}[u(x)y_1(x)] + a_0(x)u(x)y_1(x) \\ &= u'(x)y_1(x) + u(x)[y'(x) + a_0(x)y_1(x)] \\ &= u'(x)y_1(x) + 0, \end{aligned} \tag{21}$$

where we are using the fact that y_1 satisfies the associated homogeneous DE (16). Now, if (20) is indeed a particular solution, then the result of (21) requires that $u(x)$ be a solution of $u'(x)y_1(x) = f(x)$. Upon integration, we determine

$$u(x) = \int \frac{f(x)}{y_1(x)}\,dx,$$

where we can ignore the arbitrary constant of integration (i.e., any particular solution is good enough!). Hence, the particular solution is

$$\boxed{y_P = y_1(x)\int \frac{f(x)}{y_1(x)}\,dx,} \tag{22}$$

and the general solution of (15) becomes

$$\boxed{y = y_H + y_P = C_1 y_1(x) + y_1(x)\int \frac{f(x)}{y_1(x)}\,dx.} \tag{23}$$

EXAMPLE 5 Find the solution of

$$xy' + (1-x)y = xe^x, \quad y(1) = 3e.$$

Solution: We first rewrite the DE in normal form, which yields

$$y' + \left(\frac{1}{x} - 1\right)y = e^x, \quad x \neq 0.$$

Thus, using (19) we see that

$$y_1(x) = \exp\left[-\int\left(\frac{1}{x} - 1\right)dx\right] = \frac{1}{x}e^x,$$

and, consequently, the homogeneous solution (18) is $y_H = C_1 e^x/x$.

From the normal form above we note that $f(x) = e^x$, and thus, the particular solution is $y_P = u(x)e^x/x$, where

$$u(x) = \int \frac{e^x}{x^{-1}e^x}dx = \int x\,dx = \frac{x^2}{2}.$$

Hence, $y_P = \frac{1}{2}xe^x$, and the general solution we seek can be expressed in the form

$$y = y_H + y_P = \frac{C_1 e^x}{x} + \frac{1}{2}xe^x, \quad x \neq 0.$$

Last, by imposing the IC, we are led to

$$y(1) = C_1 e + \frac{1}{2}e = 3e,$$

or $C_1 = 5/2$. Thus,

$$y = \frac{1}{2}\left(\frac{5}{2x} + x\right)e^x, \quad x \neq 0.$$

1.3.3 Initial condition

When solving (15) subject to the IC (14), it can be useful for physical interpretation to split the problem into two simpler problems, defined by

PROBLEM (A): $$y' + a_0(x)y = 0, \quad y(x_0) = y_0 \tag{24}$$

PROBLEM (B): $$y' + a_0(x)y = f(x), \quad y(x_0) = 0. \tag{25}$$

If we subject the general solution (18) of the homogeneous equation in PROBLEM (A) to the prescribed IC, we are led to

$$y_H(x_0) = C_1 y_1(x_0) = y_0,$$

and hence, the IVP (24) has the unique solution

$$y_H = \frac{y_0 y_1(x)}{y_1(x_0)}. \tag{26}$$

Equation (26) physically represents the "response" of the system (15) to the IC (14) in the absence of an input function $f(x)$. Similarly, the solution of (25) is considered the response of the system when y_0 in the IC is zero, i.e., the system is "at rest." To obtain this solution, we now define the specific function

$$u(x) = \int_{x_0}^{x} \frac{f(s)}{y_1(s)}ds, \tag{27}$$

where we have introduced the dummy variable of integration s; thus,