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Coherent-Mode Representation of Optical Fields and Sources

1.1 Introduction

In the 1980s, E. Wolf proposed a new theory of partial coherence formulated in the space-frequency domain.[1,2] The fundamental result of this theory is the fact that a stationary optical field of any state of coherence may be represented as a superposition of coherent modes, i.e., elementary uncorrelated field oscillations that are spatially completely coherent.[†] The importance of this result can hardly be exaggerated since it opens a new perspective in understanding and interpreting the physics of generation, propagation, and transformation of optical radiation. In this chapter, using primarily the basic book by Mandel and Wolf,[4] we give an outline of the theory of optical coherence in the space-frequency domain and coherent-mode representations of an optical field. We also consider the concept of the effective number of modes needed for the coherent-mode representation of an optical field,[5] and give a brief survey of the known coherent-mode representations of some model sources, namely, the Gaussian Schell-model source ,[6–9] Bessel correlated source,[10] and the Lambertian source.[11]

1.2 Foundations of the Coherence Theory in the Space-Frequency Domain

Let us consider a scalar quasi-monochromatic optical field occupying some finite closed domain D. Let $V(\mathbf{r}, t)$ be the *complex analytic signal* associated with this field at a point specified by the position vector $\mathbf{r} = (x, y, z)$ and at time t. For any realistic optical field, $V(\mathbf{r}, t)$ is a fluctuating function of time, which may be regarded as a *sample realization* of some *random process*. Hence, in the general case, an optical field can only be described in statistical terms. Within the framework of the second-order moments theory of random processes, the statistical description of a fluctuating field is given by the *cross-correlation function* $\Gamma(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2)$, defined as

$$\Gamma\left(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2\right) = \left\langle V^*\left(\mathbf{r}_1, t_1\right) V\left(\mathbf{r}_2, t_2\right)\right\rangle, \tag{1.1}$$

[†] A similar result has been obtained in the past by H. Gamo in the framework of matrix treatment of partial coherence.[3]

where the asterisk denotes the complex conjugate and the angle brackets denote the average taken over an ensemble of all possible process realizations. The random field is said to be *stationary in the wide sense* if its cross-correlation function depends on the two time arguments only through their difference $\tau = t_2 - t_1$, i.e.,

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \left\langle V^*(\mathbf{r}_1, t)\, V(\mathbf{r}_2, t + \tau) \right\rangle. \tag{1.2}$$

The cross-correlation function $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ is known as the *mutual coherence function* and represents the central quantity of the *classical theory of optical coherence*. It may be noted that $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ describes an optical field in the *space-time domain*.

An alternative statistical description of an optical field may be obtained by assuming that $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ is absolutely integrable in the range $-\infty < \tau < \infty$ and, hence, may be represented by its Fourier transform

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \int_{-\infty}^{\infty} \Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) \exp(-\mathrm{i}2\pi\nu\tau)\, \mathrm{d}\tau, \tag{1.3}$$

where the Fourier variable ν has the meaning of frequency. The function $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$ is known as the *cross-spectral density function* of the field and represents the central quantity of the *coherence theory in the space-frequency domain*.

We will now note a few important properties of the cross-spectral density function. In the first place, assuming that $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$ is a continuous function of $\mathbf{r}_1$ and $\mathbf{r}_2$ bounded throughout the domain D, one necessarily finds that it is *square integrable in* D, i.e.,

$$\iint_D |W(\mathbf{r}_1, \mathbf{r}_2, \nu)|^2\, \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 < \infty. \tag{1.4}$$

In the second place, $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$ possesses *Hermitian symmetry*, i.e.,

$$W(\mathbf{r}_2, \mathbf{r}_1, \nu) = W^*(\mathbf{r}_1, \mathbf{r}_2, \nu), \tag{1.5}$$

which follows at once on taking the Fourier transform of both sides of the evident equality $\Gamma(\mathbf{r}_2, \mathbf{r}_1, -\tau) = \Gamma^*(\mathbf{r}_1, \mathbf{r}_2, \tau)$. In the third place, it may be shown (see Ref. 1, Appendix A) that $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$ is a *nonnegative definite function*, i.e.,

$$\iint_D W(\mathbf{r}_1, \mathbf{r}_2, \nu)\, f^*(\mathbf{r}_1)\, f(\mathbf{r}_2)\, \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 \geqslant 0, \tag{1.6}$$

where $f(\mathbf{r})$ is any square-integrable function.

In the particular case when $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$, the cross-spectral density function becomes the *spectral density*

$$S(\mathbf{r}, \nu) = W(\mathbf{r}, \mathbf{r}, \nu). \tag{1.7}$$

Inequality (1.6), together with definition (1.7), implies that

$$S(\mathbf{r}, \nu) \geqslant 0 \tag{1.8}$$

and

$$|W(\mathbf{r}_1, \mathbf{r}_2, \nu)| \leqslant [S(\mathbf{r}_1, \nu)]^{1/2} [S(\mathbf{r}_2, \nu)]^{1/2}. \tag{1.9}$$

In view of inequality (1.9), the normalized cross-spectral density function may be defined as

$$\mu(\mathbf{r}_1, \mathbf{r}_2, \nu) = \frac{W(\mathbf{r}_1, \mathbf{r}_2, \nu)}{[S(\mathbf{r}_1, \nu)]^{1/2} [S(\mathbf{r}_2, \nu)]^{1/2}}, \tag{1.10}$$

known as the *spectral degree of coherence*. The following relation for $\mu(\mathbf{r}_1, \mathbf{r}_2, \nu)$ is obvious:

$$0 \leqslant |\mu(\mathbf{r}_1, \mathbf{r}_2, \nu)| \leqslant 1. \tag{1.11}$$

When $|\mu| = 0$ for each pair of different points $\mathbf{r}_1$ and $\mathbf{r}_2$, the field is referred to as *completely incoherent*; when $|\mu| = 1$, as *completely coherent*; and when $0 < |\mu| < 1$, as *partially coherent* in space.

We will now consider the propagation of the cross-spectral density in *free space*, i.e., in the space that does not contain any sources or absorbers. As is well known,[4] the mutual coherence function $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ satisfies, in free space, the two *wave equations*

$$\nabla_1^2 \Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} \Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau), \tag{1.12a}$$

$$\nabla_2^2 \Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} \Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau), \tag{1.12b}$$

where $\nabla_{1(2)}^2$ is the Laplacian operator taken with respect to the point $\mathbf{r}_{1(2)}$, and c is the speed of light in a vacuum. Then, taking the Fourier transform of Eqs. (1.12) with respect to variable τ, we find that the cross-spectral density $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$ propagates in free space in accordance with the *coupled Helmholtz equations*

$$\nabla_1^2 W(\mathbf{r}_1, \mathbf{r}_2, \nu) + k^2 W(\mathbf{r}_1, \mathbf{r}_2, \nu) = 0, \tag{1.13a}$$

$$\nabla_2^2 W(\mathbf{r}_1, \mathbf{r}_2, \nu) + k^2 W(\mathbf{r}_1, \mathbf{r}_2, \nu) = 0, \tag{1.13b}$$

where $k = 2\pi\nu/c$ is the wave number. Furthermore, it will be useful to find the solution of these equations for the case when an optical field propagates into a half-space $z > 0$ with the known boundary values of cross-spectral density at all pairs of points $\mathbf{x}_1 = (x_1, y_1)$ and $\mathbf{x}_2 = (x_2, y_2)$ in the plane $z = 0$ (Fig. 1.1). The solution of Eq. (1.13b) for fixed $\mathbf{r}_1$ is given by *Rayleigh's first diffraction formula*[4] as

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = -\frac{1}{2\pi} \int_{(z=0)} W(\mathbf{r}_1, \mathbf{x}_2, \nu) \frac{\partial}{\partial z_2} \left[\frac{\exp(\mathrm{i}kR_2)}{R_2} \right] \mathrm{d}\mathbf{x}_2, \tag{1.14}$$

where $R_2 = |\mathbf{r}_2 - \mathbf{x}_2|$. The solution of Eq. (1.13a) for $\mathbf{r}_2 = \mathbf{x}_2$ is consequently given by

$$W(\mathbf{r}_1, \mathbf{x}_2, \nu) = -\frac{1}{2\pi} \int_{(z=0)} W(\mathbf{x}_1, \mathbf{x}_2, \nu) \frac{\partial}{\partial z_1} \left[\frac{\exp(-\mathrm{i}kR_1)}{R_1} \right] \mathrm{d}\mathbf{x}_1, \tag{1.15}$$

where $R_1 = |\mathbf{r}_1 - \mathbf{x}_1|$. On inserting Eq. (1.15) into Eq. (1.14), we obtain the following joint solution of Eqs. (1.13):

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \frac{1}{(2\pi)^2} \iint_{(z=0)} W(\mathbf{x}_1, \mathbf{x}_2, \nu) \times \frac{\partial}{\partial z_1} \left[\frac{\exp(-\mathrm{i}kR_1)}{R_1} \right] \frac{\partial}{\partial z_2} \left[\frac{\exp(\mathrm{i}kR_2)}{R_2} \right] \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2. \tag{1.16}$$

Calculating the derivatives in Eq. (1.16) and assuming that $(1/r_{1(2)}) \ll k$, one may readily find the following approximate expression for propagation of the cross-spectral density into the half-space:

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \left(\frac{k}{2\pi} \right)^2 \iint_{(z=0)} W(\mathbf{x}_1, \mathbf{x}_2, \nu) \times \frac{\exp[\mathrm{i}k(R_2 - R_1)]}{R_1 R_2} \cos\theta_1 \cos\theta_2 \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2. \tag{1.17}$$

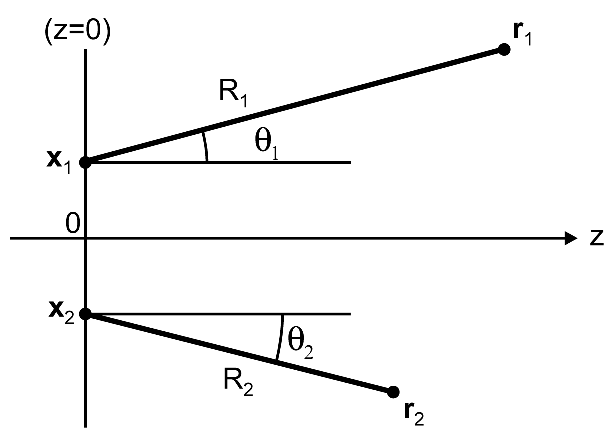

Figure 1.1 Notation relating to the propagation of the cross-spectral density function from the plane $z = 0$ into the half-space $z > 0$.

1.3 Coherent-Mode Structure of the Field

As is well known from the theory of integral equations, any continuous function that satisfies conditions (1.4)–(1.6) and, hence, the cross-spectral density $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$, may be expressed in the form of *Mercer's expansion* as

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \sum_n \lambda_n(\nu)\, \varphi_n^*(\mathbf{r}_1, \nu)\, \varphi_n(\mathbf{r}_2, \nu), \tag{1.18}$$

where $\lambda_n(\nu)$ are the *eigenvalues* and $\varphi_n(\mathbf{r}, \nu)$ are the *eigenfunctions* of the *homogeneous Fredholm integral equation of the second kind*,

$$\int_D W(\mathbf{r}_1, \mathbf{r}_2, \nu)\, \varphi_n(\mathbf{r}_1, \nu)\, \mathrm{d}\mathbf{r}_1 = \lambda_n(\nu)\, \varphi_n(\mathbf{r}_2, \nu). \tag{1.19}$$

It is important to stress that all the eigenvalues $\lambda_n(\nu)$ are real and nonnegative, i.e.,

$$\lambda_n^*(\nu) = \lambda_n(\nu) \geqslant 0, \tag{1.20}$$

and the eigenfunctions $\varphi_n(\mathbf{r}, \nu)$ are mutually orthonormal in D (if it is not already so, this may be achieved using the Gram-Schmidt procedure), i.e.,

$$\int_D \varphi_n^*(\mathbf{r}, \nu)\, \varphi_m(\mathbf{r}, \nu)\, \mathrm{d}\mathbf{r} = \delta_{nm}, \tag{1.21}$$

where δ_{nm} is the Kronecker symbol. It is appropriate to ascertain one more property of the eigenfunctions $\varphi_n(\mathbf{r}, \nu)$. On inserting Eq. (1.18) into Eq. (1.13b), we obtain

$$\sum_n \lambda_n(\nu)\, \varphi_n^*(\mathbf{r}_1, \nu)\, \nabla_2^2 \varphi_n(\mathbf{r}_2, \nu) + k^2 \sum_n \lambda_n(\nu)\, \varphi_n^*(\mathbf{r}_1, \nu)\, \varphi_n(\mathbf{r}_2, \nu) = 0. \tag{1.22}$$

Next, multiplying Eq. (1.22) by $\varphi_m(\mathbf{r}_1, \nu)$, integrating the result with respect to $\mathbf{r}_1$ over the domain D, and making use of the orthonormality relation (1.21), we find that the eigenfunctions $\varphi_n(\mathbf{r}, \nu)$ satisfy the Helmholtz equation,

$$\nabla^2 \varphi_n(\mathbf{r}, \nu) + k^2 \varphi_n(\mathbf{r}, \nu) = 0. \tag{1.23}$$

To clear up the physical meaning of expansion (1.18), we rewrite it in the form

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \sum_n \lambda_n(\nu)\, W_n(\mathbf{r}_1, \mathbf{r}_2, \nu), \tag{1.24}$$

where

$$W_n(\mathbf{r}_1, \mathbf{r}_2, \nu) = \varphi_n^*(\mathbf{r}_1, \nu)\, \varphi_n(\mathbf{r}_2, \nu). \tag{1.25}$$

It follows directly from Eqs. (1.23) and (1.25) that the function $W_n(\mathbf{r}_1, \mathbf{r}_2, \nu)$ satisfies the equations

$$\nabla_1^2 W_n(\mathbf{r}_1, \mathbf{r}_2, \nu) + k^2 W_n(\mathbf{r}_1, \mathbf{r}_2, \nu) = 0, \tag{1.26a}$$

$$\nabla_2^2 W_n(\mathbf{r}_1, \mathbf{r}_2, \nu) + k^2 W_n(\mathbf{r}_1, \mathbf{r}_2, \nu) = 0, \tag{1.26b}$$

which are just the same as those governing the free-space propagation of the cross-spectral density $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$. Hence, the function $W_n(\mathbf{r}_1, \mathbf{r}_2, \nu)$ may be regarded as the cross-spectral density associated with a *mode* of the field. Next, making use of Eqs. (1.10) and (1.25), we find that the spectral degree of coherence of each field mode is given by

$$\mu_n(\mathbf{r}_1, \mathbf{r}_2, \nu) = \frac{\varphi_n^*(\mathbf{r}_1, \nu)\, \varphi_n(\mathbf{r}_2, \nu)}{|\varphi_n(\mathbf{r}_1, \nu)|\, |\varphi_n(\mathbf{r}_2, \nu)|}. \tag{1.27}$$

It follows from Eq. (1.27) that

$$|\mu_n(\mathbf{r}_1, \mathbf{r}_2, \nu)| = 1, \tag{1.28}$$

i.e., that each field mode represents the spatially completely coherent contribution. Thus, *expansion (1.24) may be interpreted as representing the cross-spectral density of the field as a superposition of contributions from modes that are completely coherent in the space-frequency domain*. For this reason, we will refer to expansion (1.18) as the *coherent-mode representation of the field*. We will also refer to the set

$$\Lambda = \{\lambda_n(\nu), \varphi_n(\mathbf{r}, \nu)\} \tag{1.29}$$

as the *coherent-mode structure of the field*. In the special case when the integral equation (1.19) admits only one solution $\varphi(\mathbf{r}, \nu)$ associated with an eigenvalue $\lambda(\nu)$, Eq. (1.18) takes the form

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \lambda(\nu)\, \varphi^*(\mathbf{r}_1, \nu)\, \varphi(\mathbf{r}_2, \nu), \tag{1.30}$$

which implies that the field consists of the sole coherent mode, i.e., that it is spatially completely coherent at frequency ν.

Equation (1.18) allows us to obtain some other useful coherent-mode representations. Indeed, on making use of representation (1.18) in definition (1.7), we obtain the relation

$$S(\mathbf{r}, \nu) = \sum_n \lambda_n(\nu)\, |\varphi_n(\mathbf{r}, \nu)|^2. \tag{1.31}$$

On integrating Eq. (1.31) over D with due regard for Eq. (1.21), we come to the relation

$$\int_D S(\mathbf{r}, \nu)\, d\mathbf{r} = \sum_n \lambda_n(\nu). \tag{1.32}$$

On making use of definition (1.25) and Eq. (1.21), we obtain the following orthonormality relation:

$$\iint_D W_n^*(\mathbf{r}_1,\mathbf{r}_2,\nu)\, W_m(\mathbf{r}_1,\mathbf{r}_2,\nu)\, \mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2 = \delta_{nm}. \tag{1.33}$$

Finally, applying the relation

$$|W(\mathbf{r}_1,\mathbf{r}_2,\nu)|^2 = \sum_n\sum_m \lambda_n(\nu)\,\lambda_m(\nu)\, W_n^*(\mathbf{r}_1,\mathbf{r}_2,\nu)\, W_m(\mathbf{r}_1,\mathbf{r}_2,\nu), \tag{1.34}$$

obtained directly from definition (1.25), and integrating its both sides twice over the domain D with due regard for relation (1.33), we find that

$$\iint_D |W(\mathbf{r}_1,\mathbf{r}_2,\nu)|^2\, \mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2 = \sum_n \lambda_n^2(\nu). \tag{1.35}$$

The deduced modal relations (1.31), (1.32), and (1.35), as well as the basic coherent-mode representation (1.18), will be widely used in our subsequent considerations.

1.4 Ensemble Representation of the Cross-Spectral Density Function

On making use of the coherent-mode representation (1.18), one may deduce another useful representation of the cross-spectral density function expressed in terms of the ensemble of field realizations.

Let us construct a random function of the form

$$U(\mathbf{r},\nu) = \sum_n a_n(\nu)\,\varphi_n(\mathbf{r},\nu), \tag{1.36}$$

where $\varphi_n(\mathbf{r},\nu)$ are, as before, the eigenfunctions of Eq. (1.19) and $a_n(\nu)$ are some random variables that will be specified later. Since, as follows from Eq. (1.23), each term in expansion (1.36) satisfies the Helmholtz equation, the function $U(\mathbf{r},\nu)$ does the same, i.e.,

$$\nabla^2 U(\mathbf{r},\nu) + k^2 U(\mathbf{r},\nu) = 0. \tag{1.37}$$

Hence, the function $U(\mathbf{r},\nu)$ may be considered as an *optical signal*, i.e., the time-independent part of a monochromatic wave function

$$V(\mathbf{r},t) = U(\mathbf{r},\nu)\exp(-\mathrm{i}2\pi\nu t). \tag{1.38}$$

The cross-correlation function of the optical signal (1.36) at two points $\mathbf{r}_1$ and $\mathbf{r}_2$ is given by

$$\left\langle U^*(\mathbf{r}_1,\nu)\, U(\mathbf{r}_2,\nu)\right\rangle = \sum_n \sum_m \left\langle a_n^*(\nu)\, a_m(\nu)\right\rangle \varphi^*(\mathbf{r}_1,\nu)\, \varphi(\mathbf{r}_2,\nu)\,, \tag{1.39}$$

where the angle brackets, unlike those used in Eq. (1.1), this time denote the statistical averaging over an ensemble of frequency-dependent (not time-dependent) realizations.

Let us now assume that the random variables $a_n(\nu)$ are chosen to satisfy the condition

$$\left\langle a_n^*(\nu)\, a_m(\nu)\right\rangle = \lambda_n(\nu)\, \delta_{nm}\,, \tag{1.40}$$

where $\lambda_n(\nu)$ are, as before, the eigenvalues of Eq. (1.19). The condition (1.40) can be satisfied, for example, by taking

$$a_n(\nu) = [\lambda_n(\nu)]^{1/2} \exp(\mathrm{i}\theta_n)\,, \tag{1.41}$$

where θ_n are statistically independent random variables uniformly distributed in the interval $[0, 2\pi]$. Applying condition (1.40) to Eq. (1.39), we obtain

$$\left\langle U^*(\mathbf{r}_1,\nu)\, U(\mathbf{r}_2,\nu)\right\rangle = \sum_n \lambda_n(\nu)\, \varphi_n^*(\mathbf{r}_1,\nu)\, \varphi_n(\mathbf{r}_2,\nu)\,. \tag{1.42}$$

Finally, comparing Eqs. (1.42) and (1.18), we come to a new representation of the cross-spectral density function in the form

$$W(\mathbf{r}_1,\mathbf{r}_2,\nu) = \left\langle U^*(\mathbf{r}_1,\nu)\, U(\mathbf{r}_2,\nu)\right\rangle. \tag{1.43}$$

This ensemble representation may be considered as *an alternative definition of the cross-spectral density function* $W(\mathbf{r}_1,\mathbf{r}_2,\nu)$ *in the form of the cross-correlation function of the optical signal given by* Eq. (1.36) *with condition* (1.40). Applying this definition, we may obtain a new representation of the spectral density $S(\mathbf{r},\nu)$,

$$S(\mathbf{r},\nu) = \left\langle |U(\mathbf{r},\nu)|^2\right\rangle. \tag{1.44}$$

This representation clearly shows that spectral density represents the spatial distribution of an average squared modulus of monochromatic oscillations and, hence, $S(\mathbf{r},\nu)$ may be referred to as the *power spectrum* of an optical field.

1.5 Effective Number of Coherent Modes

We will inquire now about the number of coherent modes needed to represent a random field in D. To do this, we use the concept of the *effective number of coherent modes* introduced in Ref. 5.

As follows from Section 1.3, the eigenvalues $\lambda_n(\nu)$ may be arranged in a non-increasing sequence as

$$\lambda_0(\nu) \geqslant \lambda_1(\nu) \geqslant \lambda_2(\nu) \geqslant \cdots \geqslant \lambda_n(\nu) \geqslant \cdots \geqslant 0. \tag{1.45}$$

Hence, one may equate each of the lowest-order eigenvalues in Eq. (1.32) with $\lambda_0(\nu)$, and take the rest to be equal to zero. This allows the following definition of the effective number $\mathcal{N}(\nu)$ of coherent modes needed to represent the field:

$$\mathcal{N}(\nu) \equiv \frac{1}{\lambda_0(\nu)} \sum_{n=0}^{\infty} \lambda_n(\nu). \tag{1.46}$$

As can be seen, the number $\mathcal{N}(\nu)$ is, in general, noninteger; but for convenience, in practice it may be approximated by its integer part. It is obvious that the number $\mathcal{N}(\nu)$ depends on the statistical properties of the field. To estimate its upper bound, we use the inequality

$$\sum_{n=0}^{\infty} \left(\frac{\lambda_n(\nu)}{\lambda_0(\nu)} \right)^2 \leqslant \sum_{n=0}^{\infty} \frac{\lambda_n(\nu)}{\lambda_0(\nu)}, \tag{1.47}$$

which is true in view of relation (1.45). From this inequality we obtain a lower bound on the value $\lambda_0(\nu)$ as

$$\lambda_0(\nu) \geqslant \frac{\sum\limits_{n=0}^{\infty} \lambda_n^2(\nu)}{\sum\limits_{n=0}^{\infty} \lambda_n(\nu)}. \tag{1.48}$$

On making use of Eqs. (1.48) and (1.46), we find the upper bound on the number $\mathcal{N}(\nu)$ to be

$$\mathcal{N}(\nu) \leqslant \frac{\left(\sum\limits_{n=0}^{\infty} \lambda_n(\nu) \right)^2}{\sum\limits_{n=0}^{\infty} \lambda_n^2(\nu)}. \tag{1.49}$$

Finally, to express the upper bound on the effective number of coherent modes needed to represent the field in terms of the cross-spectral density, we apply the modal relations (1.32) and (1.35) into Eq. (1.49) to obtain

$$\mathcal{N}(\nu) \leqslant \frac{\left(\int_D S(\mathbf{r},\nu)\,\mathrm{d}\mathbf{r}\right)^2}{\iint_D |W(\mathbf{r}_1,\mathbf{r}_2,\nu)|^2\,\mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2}. \tag{1.50}$$

To clarify the physical meaning of the obtained result, in Ref. 5 the following definitions of the *effective volume of the field* and the *effective coherence volume* are introduced, respectively:

$$\mathcal{V}_{\mathrm{e}}(\nu) = \frac{1}{S_{\max}(\nu)} \int_D S(\mathbf{r},\nu)\,\mathrm{d}\mathbf{r}, \tag{1.51}$$

$$\mathcal{V}_{\mathrm{ce}}(\nu) = \frac{1}{\mathcal{V}_{\mathrm{e}}(\nu) S_{\max}^2(\nu)} \iint_D |W(\mathbf{r}_1,\mathbf{r}_2,\nu)|^2\,\mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2, \tag{1.52}$$

where

$$S_{\max}(\nu) = \max_{\mathbf{r}\in D} S(\mathbf{r},\nu). \tag{1.53}$$

By applying definitions (1.51) and (1.52) into Eq. (1.50), we obtain

$$\mathcal{N}(\nu) \leqslant \frac{\mathcal{V}_{\mathrm{e}}(\nu)}{\mathcal{V}_{\mathrm{ce}}(\nu)}. \tag{1.54}$$

Thus, *the more incoherent is the field, the more coherent modes are needed for its representation*.

Concluding this section, we note that the effective number $\mathcal{N}(\nu)$ of coherent modes may be used in practice to establish an optimal point for truncating the modal representation (1.18).

1.6 Coherent-Mode Representations of Some Model Sources

The mode representation of the field considered in Section 1.3 may be applied without any changes for describing the optical source, which can be a primary or a secondary one. Furthermore, this representation may be used for many infinite sources. To find the coherent-mode structure of the source, it is necessary to solve the integral equation (1.19) with the kernel given by the cross-spectral density $W(\mathbf{r}_1,\mathbf{r}_2,\nu)$ of the true source distribution (in the case of a primary source) or the field distribution across the source (in the case of a secondary source). Unfortunately, the solutions of this equation in a closed form are obtained at present only for a very limited number of source models. A brief review of the main known solutions of the integral equation (1.19) is given below.