2.2.6 Newtonian Imaging Equation

In the Gaussian imaging equation (2-4), the object and image distances S and S', respectively, are measured from the vertex V of the refracting surface. In the Newtonian imaging equation, they are measured from the respective focal points F and F'. Thus, let z and z' be the object and image distances from the focal points F and F', respectively, as indicated in Figure 2-14. From similar triangles P_0FP and FVA in this figure, we note that the transverse magnification may be written

$$\boxed{M_t \equiv h'/h = -f/z \quad ,} \tag{2-22}$$

where z (like f) is numerically negative because P_0 lies to the left of the reference point F. Similarly, from similar triangles $VF'B$ and $F'P_0'P'$, it may also be written

$$\boxed{M_t = -z'/f' \quad .} \tag{2-23}$$

Equating the right-hand sides of these equations, we obtain the *Newtonian imaging equation*:

$$\boxed{zz' = ff' = -(n/n')f'^2 \quad .} \tag{2-24}$$

It is evident from Eq. (2-24) that z and z' must have opposite signs, implying that an object and its image lie on the opposite sides of the corresponding focal points. For example, if the object lies to the left of F, then the image lies to the right of F'. Differentiating both sides of Eq. (2-24) and using Eqs. (2-8), (2-22), and (2-23), we obtain Eq. (2-18), relating the longitudinal and transverse magnifications.

2.3 THIN LENS

It should be evident that the image formed by a lens consisting of two refracting surfaces can be obtained by a repeated application of the imaging equation for a refracting surface. The image formed by the first surface becomes the object for the second, and its image by the second surface yields the image formed by the lens. In this section, we consider imaging by a *thin lens* in air, i.e., one for which the spacing between its two surfaces is negligible. We derive simple imaging equations for a thin lens such that it is not necessary to apply the imaging equations for each surface to determine the image of an object. Thus, we show that it is possible to determine the image of an object without determining the image formed by its two surfaces sequentially. The imaging equations when the media on the two sides of a thin lens are different, are also given. Finally, it is shown that the power a system consisting of thin lenses in contact is equal to the sum of the powers of the individual lenses.

2.3.1 Gaussian Imaging Equation

Consider a thin lens in air made of a material of refractive index n, as illustrated in Figure 2-16. Let the radii of curvature of its two surfaces be R_1 and R_2, with their centers of curvature at C_1 and C_2, respectively. The line joining C_1 and C_2 defines the

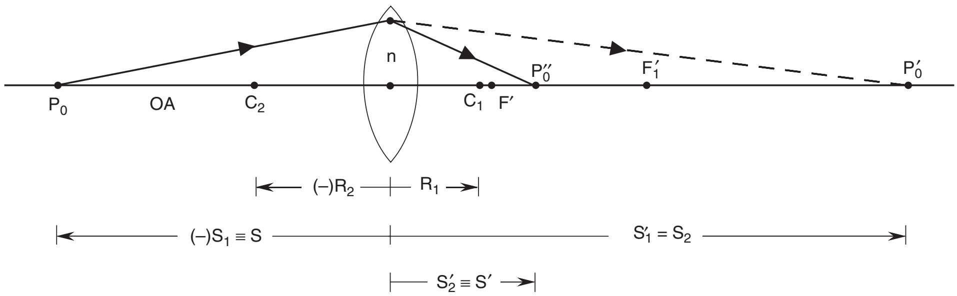

Figure 2-16. Imaging of an axial point object P_0 by a thin lens of refractive index *n*. The lens surfaces have radii of curvature of R_1 and R_2. The line *OA* connecting their centers of curvature C_1 and C_2 defines the optical axis of the lens. *C* is the center of the lens. P_0' is the image of P_0 formed by the first surface, and P_0'' is the image of the virtual object P_0' formed by the second surface.

optical axis *OA* of the lens. Consider an axial point object P_0 lying at a distance S_1 from the lens. Its image P_0' formed by the first surface lies at a distance S_1' that, according to Eq. (2-4), is given by

$$\frac{n}{S_1'} - \frac{1}{S_1} = \frac{n-1}{R_1} \quad . \tag{2-25}$$

A ray from P_0 is refracted by the surface intersecting the optical axis at P_0'. This image is a virtual object for the second surface because the rays associated with it appear to converge to it rather than actually diverge from it. It lies at a distance $S_2 = S_1'$. Its image P_0'' formed by the surface lies at a distance S_2', that, according to Eq. (2-4), is given by

$$\frac{1}{S_2'} - \frac{n}{S_1'} = \frac{1-n}{R_2} \quad . \tag{2-26}$$

Adding Eqs. (2-24) and (2-25), we obtain

$$\frac{1}{S'} - \frac{1}{S} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad , \tag{2-27}$$

where we have let $S_1 = S$ and $S_2' = S'$ be the object and final image distances, as indicated in Figure 2-16. Equation (2-27) is the Gaussian imaging equation relating the object and image distances.

2.3.2 Focal Lengths and Refracting Power

By definition, *image-space focal length* f' represents the image distance when the object lies at infinity, i.e., $S' = f'$ when $S = -\infty$. Therefore, from Eq. (2-27), f' is given by

$$\boxed{\frac{1}{f'} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)} \quad . \tag{2-28}$$

Thus, a ray incident on the lens parallel to its optical axis is refracted by the first surface intersecting the optical axis at F_1' at a distance $nR_1/(n-1)$, as illustrated in Figure 2-16. This ray is refracted by the second surface intersecting the optical axis at F', which is the *image-space focal point.* In effect, the parallel ray incident on the lens is refracted by it passing through F', as illustrated in Figure 2-17a. Similarly, by definition of the object-space focal length, *f* represents the object distance that yields an image at infinity. Thus, $S' = \infty$ when $S = f$, where $f = -f'$. A ray from the *object-space focal point F* incident on the lens emerges from it parallel to its optical axis upon refraction, as illustrated in Figure 2-17b. It should be evident that the focal points *F* and F', which lie on the opposite sides of the lens, are not conjugates of each other. The imaging equation (2-27) can be written in terms of the focal length f' as

$$\frac{1}{S'} - \frac{1}{S} = \frac{1}{f'} \quad . \tag{2-29}$$

The right-hand side of Eq. (2-27) represents the *refracting power K* of the lens. Its reciprocal is called the *equivalent* or *effective focal length* f_e of the lens. Thus, we may write

$$K = \frac{1}{f_e} = \frac{1}{f'} \tag{2-30}$$

$$= (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{2-31a}$$

$$= (n-1)\,(C_1 - C_2) \quad , \tag{2-31b}$$

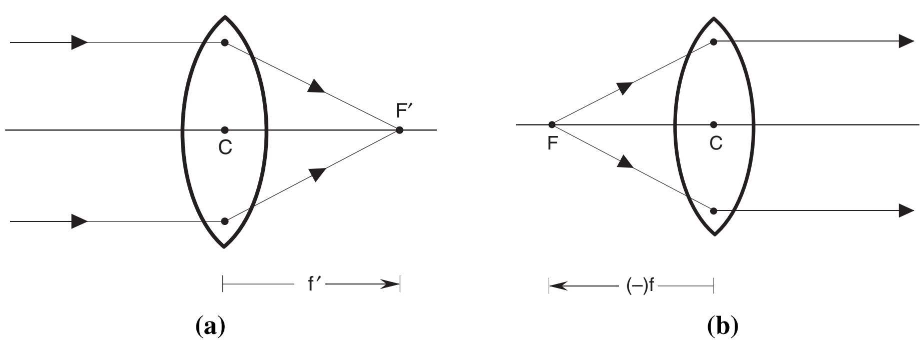

Figure 2-17. Focal points of a positive thin lens with its center *C*. (a) Image-space focal point F'. (b) Object-space focal point *F*. Both focal points are real in that parallel rays converge to F', and rays actually originating from *F* form a parallel beam after refraction by the lens.

where $C = 1/R$ is the curvature of a surface. We note that the refracting power of the lens is equal to the sum of the refracting powers K_1 and K_2 of its two surfaces, i.e.,

$$K = K_1 + K_2 \quad , \tag{2-32}$$

where

$$K_1 = \frac{n-1}{R_1} \tag{2-33a}$$

and

$$K_2 = \frac{1-n}{R_2} \quad . \tag{2-33b}$$

We note that the focal length or the power of a lens depends on the difference in the curvatures of its surfaces but not on the curvatures themselves. Thus, if the curvatures of the lens surfaces are changed by the same amount, its shape changes without changing its Gaussian properties. This degree of freedom, called the *bending* of the lens, is used in reducing its aberrations. The equation (2-31) for the focal length of a thin lens, in terms of its refractive index and the curvatures of its surfaces, has traditionally been called the *lens maker's formula*. This is, however, not correct because a lens of zero thickness cannot be fabricated. This name should instead be associated with Eq. (4-41) for a thick lens (described in Chapter 4).

A lens with a positive value of K, f_e, or f', as illustrated in Figure 2-17, is called a *converging* or a *positive lens*. A lens with the curvatures of its two surfaces having the same magnitude but opposite signs is referred to as an *equiconvex lens*. The surfaces refract a ray incident on the lens toward the optical axis. Similarly, a lens with a negative value of K, f_e, or f' is called a *diverging* or a *negative lens*.. It is shown in Figure 2-18, illustrating its focal points. Parallel rays incident on the lens, as in Figure 2-18a, are refracted by it, appearing to diverge from the image-space focal point F'. Similarly, rays converging to the virtual object-space focal point F are refracted by the lens into a parallel beam, as illustrated in Figure 2-18b. A lens whose first surface has a negative curvature and second surface has a positive curvature of the same magnitude as the first is referred to as an *equiconcave lens.*

A lens with surface curvatures of the same sign is called a *meniscus lens*. It can be positive or negative, as illustrated in Figure 2-19. Unless it is surrounded by a medium of higher refractive index, a lens that is thick at the center compared to its edges is positive, and a lens that is thin at the center is negative. Of course, one of the surfaces may be planar, in which case the lens is called planoconvex or planoconcave, depending on the curvature of the other surface.

It should be noted, however, that if a beam converging to F' is incident on a positive lens, as in Figure 2-20a, i.e., a virtual point object P_0 at F', a real image is formed at P_0'.

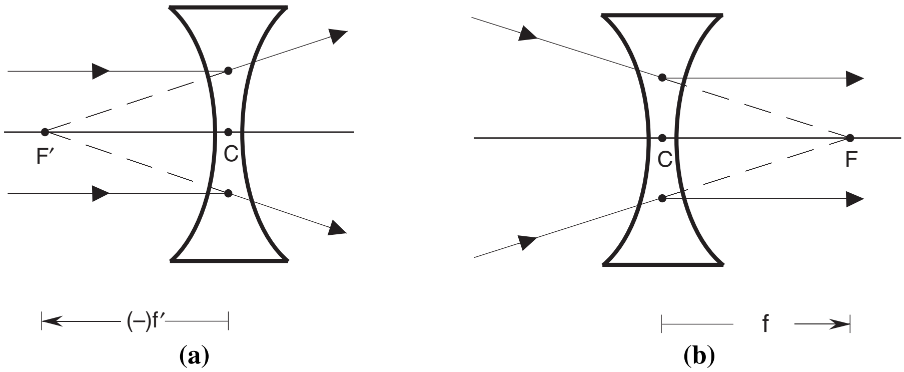

Figure 2-18. Focal points of a negative thin lens. (a) Image-space focal point F'. (b) Object-space focal point F. Both focal points are virtual in that parallel rays appear to diverge from F', or rays appearing to converge to F form a parallel beam after refraction by the lens.

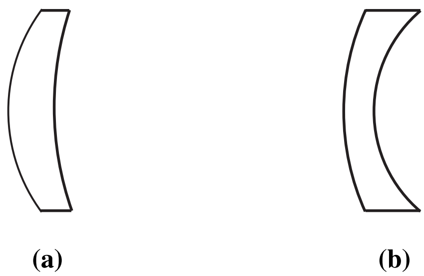

Figure 2-19. (a) Positive and (b) negative meniscus lens. The radii of curvature of their surfaces have the same sign. The lens thickness at the center is higher compared to that at the edges for a positive meniscus, and lower for a negative meniscus.

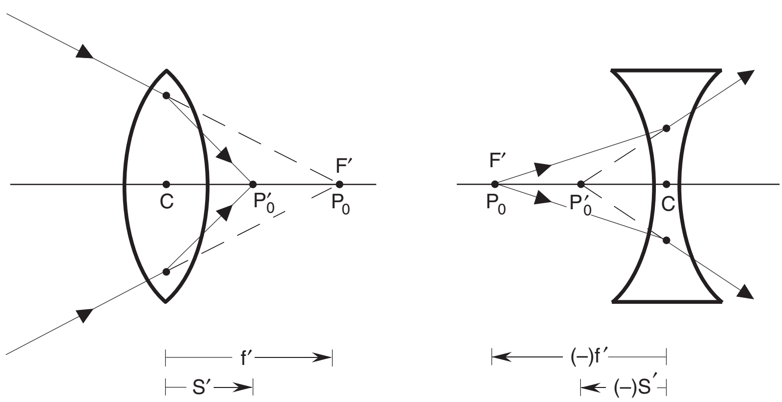

Figure 2-20. Virtual point object P_0 at the real focus F' of a positive lens. The image point P_0' is real. (b) Real point object P_0 at the virtual focus F of a negative lens. The image point P_0' is virtual.

Similarly, if a point object is placed at the focal point F' of a negative lens, as in Figure 2-20b, i.e., a real point object P_0 at F', a virtual image is formed at P_0'. The image distance in both cases is given by half the corresponding focal length.

2.3.3 Magnifications and Lagrange Invariant

The transverse magnification of the image formed by the lens can be obtained by applying Eq. (2-12) to the images formed by its two surfaces. A ray from an off-axis point object P passing through the center of curvature C_1 of the first surface is shown in Figure 2-21 intersecting the image plane at its image P'. The magnification of the inverted image $P_0'P'$ of the object P_0P formed by the first surface is given by

$$M_1 \equiv h_1' / h_1 \tag{2-34a}$$

$$= \frac{S_1'}{nS_1} \quad . \tag{2-34b}$$

A parallel ray from P is refracted by the first surface passing through its focal point F_1', which, in turn, is refracted by the second surface passing through the focal point F' of the lens and intersecting the final image plane at the image point P''. The magnification of the erect image $P_0''P''$ of the object $P_0'P'$ formed by the second surface is given by

$$M_2 \equiv h_2' / h_2 = h_2' / h_1' \tag{2-35a}$$

$$= \frac{n_l S_2'}{S_1'} \quad . \tag{2-35b}$$

Therefore, the transverse magnification of the final image $P_0''P''$ of the object P_0P formed by the lens as a whole is given by

$$M_t = M_1 M_2 = h_2'/h_1 = S_2'/S_1 \quad , \tag{2-36}$$

or

$$\boxed{M_t \equiv h'/h = S'/S \quad ,} \tag{2-37a}$$

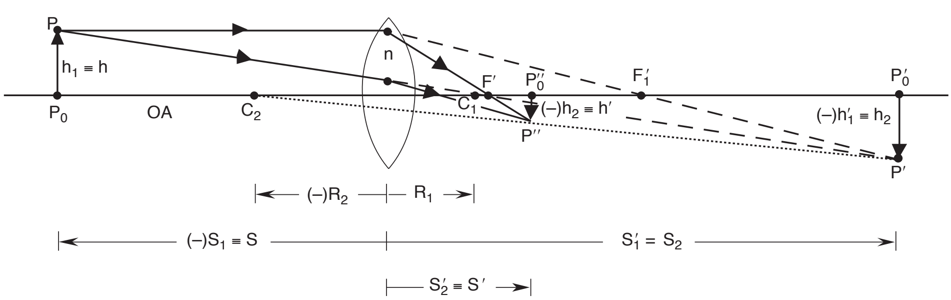

Figure 2-21. Imaging of an off-axis point object P. The dotted line simply shows that the final image P'' lies on the line joining P' and C_2, as expected.

where we have let $h = h_1$ and $h' = h_2'$ be the object and final image heights, respectively. Substituting for S' from Eq. (2-29) into Eq. (2-37a), the magnification can also be written in terms of S and f':

$$M_t = \frac{f'}{f' + S} \quad . \tag{2-37b}$$

The *angular magnification* of a ray bundle diverging from the axial point object P_0 and converging toward its image P_0' (see Figure 2-22) is given by

$$\boxed{M_\beta = \beta_0'/\beta_0 = S/S' \quad .} \tag{2-38}$$

From Eqs. (2-37) and (2-38), we find that the product of the transverse magnification of the image and the angular magnification of the ray bundle for a thin lens is given by

$$\boxed{M_t M_\beta = 1 \quad .} \tag{2-39}$$

From the definitions of the magnifications, Eq. (2-39) can also be written

$$h'\beta_0' = h\beta_0 \quad , \tag{2-40}$$

showing that the quantity $h\beta_0$ is invariant upon refraction by the lens. This quantity is called the *Lagrange invariant*. [It is shown in Section 5.4.10 that the object flux entering the lens is proportional to its square.] From Eq. (2-40), the transverse magnification of the image can also be written

$$\boxed{M_t = \beta_0/\beta_0' \quad ,} \tag{2-41}$$

i.e., it is given by the ratio of the slope angles of the incident and refracted rays for an axial point object.

Differentiating both sides of Eq. (2-27), we obtain the longitudinal magnification of the image:

$$\boxed{M_l \equiv \Delta S'/\Delta S = \left(S'/S\right)^2 = M_t^2 = M_t/M_\beta \quad .} \tag{2-42}$$

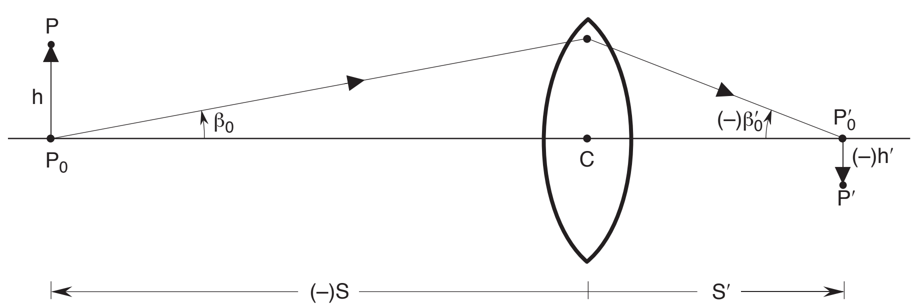

Figure 2-22. Lagrange invariant $h\beta_0$ for imaging by a thin lens.

The comments made following Eq. (2-18) apply to Eq. (2-42) as well. Thus, for example, when the object is displaced longitudinally, the image is displaced in the same direction as the object. In Eq. (2-42), the lens is assumed to be fixed in position, and $\Delta S'$ represents the displacement of the image corresponding to a displacement ΔS of the object. However, if the object is fixed and the lens is displaced by an amount Δ, then the corresponding displacement of the image is $\left(1 - M_t^2\right)\Delta$, as shown in Section 2.9.3.

2.3.4 Graphical Imaging

The Gaussian image of a point object can be located graphically, as illustrated in Figure 2-23, in the same manner as in Section 2.2.5 for the case of a refracting surface, except that a ray through the center of curvature of the surface is replaced by one through the center of the lens. Thus, a ray from an object point P incident parallel to the optical axis of the lens emerges from it passing through its image-space focal point F', and a ray incident in the direction of its object-space focal point F emerges parallel to the optical axis. The intersection of these two rays locates the image point P'. The ray passing through F determines the image height h'. The transverse magnification given by Eq. (2-37) dictates similarity of the triangles P_0CP and $P_0'CP'$, showing that a ray incident in the direction of the center C of the lens passes through it undeviated. Figure 2-23 is similar to Figure 2-16 except that the two-step imaging (one for each surface) has been replaced by single-step imaging. Only two of the three rays from an off-axis point object, namely, parallel to the axis, in the direction of the object-space focal point, and in the direction of the center, are needed to determine the image point. Of course, the third ray provides a good check on the correctness of the drawing.

2.3.5 Newtonian Imaging Equation

In the Gaussian imaging equation (2-27), the object and image distances S and S', respectively, are measured from the lens center. In the corresponding *Newtonian imaging equation,* they are measured from the respective focal points. Thus, as indicated in Figure 2-23, let z and z' be the object and image distances from the focal points F and F', respectively. From similar triangles P_0FP and FCA, we note that the transverse magnification of the image can be written

$$M_t \equiv h'/h = -f/z \quad . \tag{2-43}$$

Similarly, from similar triangles $CF'B$ and $P_0'F'P'$, it may also be written

$$\boxed{M_t = -z'/f' \quad .} \tag{2-44}$$

The negative sign on the right-hand sides of Eqs. (2-43) and (2-44) has been introduced because M_t in Figure 2-23 is numerically negative due to h' being numerically negative. From Eqs. (2-47) and (2-44), we obtain

$$\boxed{zz' = ff' = -f'^2 \quad ,} \tag{2-45}$$