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Chapter 7
Polarized Form of the Governing Equation Including Atmospheric Scattering Terms

In this chapter we will look at all the energy sources that contribute to the polarimetric radiance reaching the sensor. In Sec. 7.1 we introduce the radiometric terms of interest in the form of a governing equation. To better understand the polarimetric nature of these terms, we delve into the polarized behavior of atmospheric scattering in Sec. 7.2. The radiometric terms and concepts used throughout this section draw heavily on the radiometry fundamentals introduced in Chapter 2.

7.1 Governing Polarized Radiance Equation

In order to analyze remotely sensed polarimetric data, we need to develop a governing equation describing all the terms that contribute to the polarimetric radiance reaching the sensor. First, the radiometric equation for the unpolarized radiance is introduced and nomenclature is established. Then, the polarimetric representation of these equations is derived. This representation highlights the role of the polarimetric BRDF and guides a simplified field measurement technique introduced in Chapter 10. The polarized radiance governing equation forms the basis for all subsequent remote sensing studies. The approach presented here follows that of Shell and Schott (2005).

7.1.1. Scalar representation of the governing equation

The total radiance in the visible to near infrared (VNIR) portion of the spectrum (i.e., that of solar origin) reaching a sensor aperture (L_s) may be approximated as the sum of three radiance sources (see Schott (2007) for a more complete derivation of the terms in the governing equation):

(1) direct solar reflection from the target (L_r)
(2) *upwelled* atmospheric *radiance* resulting from solar scatter along the target to sensor path (L_u) and
(3) target-reflected *downwelled* radiance from the skydome (L_d).

The order of the radiance terms above is that of typically decreasing magnitude, though the ground or target reflectance and atmospheric conditions greatly influence their relative values (cf. Fig. 4.12 and Table 4.1 of Schott (2007)). These radiance terms are functions of the incident and reflected zenith angles (θ_i, θ_r) and reflected azimuth angle (ϕ).

An expression for the radiance from the direct solar reflection (L_r) is obtained by first considering the *exoatmospheric solar irradiance* (E_s) that propagates through the atmosphere along the solar-to-target path having a transmittance of τ_i. When incident upon a surface, it is then reflected, and again attenuated by the atmosphere along the ground-to-sensor atmospheric path by τ_r. We can use the bidirectional reflectance distribution function (f_r) introduced in Chapter 6 to relate the reflected radiance to the incident irradiance. Assembling these terms, (L_r) may therefore be expressed as

$$L_r = \tau_r(\theta_r) f_r(\theta_i, \theta_r, \phi) \cos\theta_i \, \tau_i(\theta_i) E_s(\theta_i). \tag{7.1}$$

Care must be used with the coordinate systems in Eq. (7.1). BRDF is defined relative to the material surface normal, which generally is not coincident with the zenith direction. This requires rotation of the coordinate systems to account for the local surface normal's deviation from the normal to the earth. However, for purposes of introducing this approach, we consider only zenith facing materials such that Eq. (7.1) is adequate. (Secondary illumination from adjacent surfaces and shadowing are also important, but are considered secondary effects and not necessary for this discussion.)

In a similar fashion, target-reflected radiance from the sky (L_d) may be derived. The downwelled radiance distributed over the entire sky hemisphere ($L_d^{\Omega i}$) is integrated to sum irradiance contributions onto the target from the sky, which is modified by the cosine of the incident angle from the surface normal. As before, each of these irradiance contributions is then reflected by the surface BRDF, which is then attenuated by the target-to-sensor atmospheric transmittance. Replacing the BRDF by an isotropic reflectance factor not having angular dependency greatly simplifies the expression, as the reflectance factor may be placed outside the integral. However, the more stringent BRDF must be retained, as it is essential to polarimetry. An appropriate expression for L_d is therefore

$$L_d = \tau_r(\theta_r) \iint\limits_{\Omega_i} f_r(\theta_i, \theta_r, \phi) \, \cos\theta_i \, L_d^{\Omega_i}(\theta_i, \phi) \, d\Omega_i, \tag{7.2}$$

where $d\Omega_i = \sin\theta_i \, d\theta_i d\phi$[sr].

A full representation for the upwelled atmospheric radiance (L_u) will not be attempted, as it is rather complex and usually approximated by atmospheric scattering codes such as *MODTRAN* [Berk et al. (1999)], as is the downwelled sky radiance component ($L_d^{\Omega i}$) in Eq. (7.2). The upwelled radiance is given simply to show the geometry dependence as

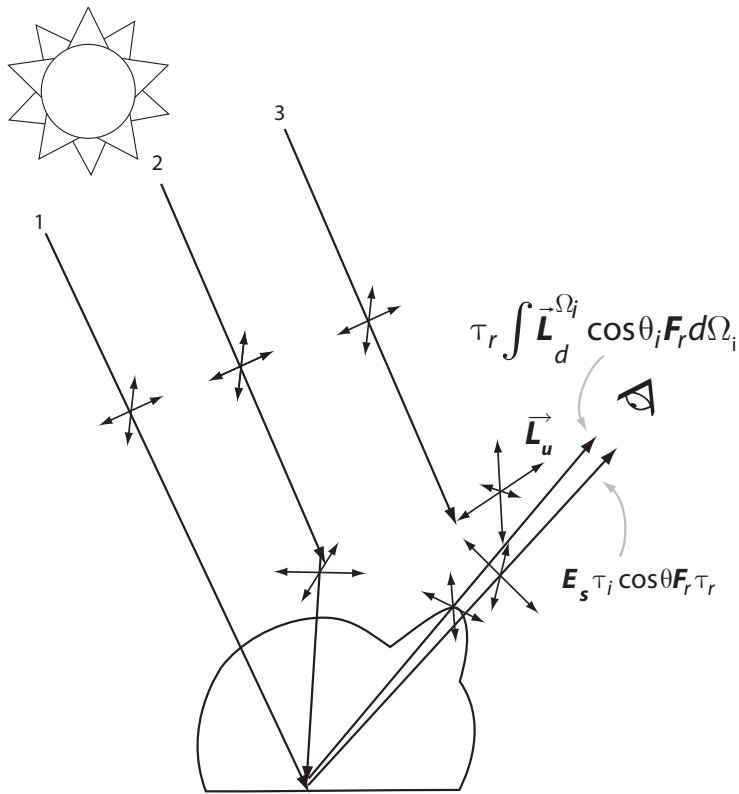$$L_u = L_u(\theta_r, \phi). \tag{7.3}$$

Figure 7.1 Illustration of key terms and energy paths in the governing equation. Path one shows the randomly polarized incident flux bidirectionally reflected and polarized by the target and propagated to the sensor. Path two shows randomly polarized solar illumination scattered and polarized by the atmosphere and propagated to the target where it is polarimetrically bidirectionally reflected, integrated over the hemisphere above the target, and propagated to the sensor. Path three shows randomly polarized solar illumination scattered and polarized by the atmosphere and propagated to the sensor.

7.1.2 Governing equation—Stokes representation

Transforming Eqs. (7.1) through (7.3) into the polarized representation is accomplished using the Mueller-Stokes formalism introduced in Chapters 4 and 5. In brief, all radiometric flux values are replaced by Stokes vectors and "transfer" functions such as atmospheric transmittance and reflectance (BRDF) are replaced by Mueller matrices [Shell and Schott (2005)]. Prior to making these substitutions, some simplifications are appropriate (see Fig. 7.1). First, the exoatmospheric solar irradiance may be considered randomly polarized, so only the scalar magnitude (or first Stokes component) of the direct solar irradiance needs to be considered. Second, the atmospheric transmittance values in Eqs. (7.1) through (7.3) all primarily represent absorption of forward scattering, which retains the incident polarization. Therefore, the scalar values for τ_i and τ_r may be used without resorting to a Mueller matrix representation. Eqs. (7.1) through (7.3) therefore become

$$\vec{L}_r = \tau_r(\theta_r) \; \mathbf{F}_r \; (\theta_i, \theta_r, \phi) \tau_i(\theta_i) \; \cos\theta_i \; E_s(\theta_i), \tag{7.4}$$

$$\vec{L}_d \; = \; \tau_r(\theta_r) \iint\limits_{\Omega_i} \mathbf{F}_r(\theta_i, \theta_r, \phi) \; \cos\theta_i \; \vec{L}_d^{\Omega_i} \; (\theta_i, \phi_i) \; d\Omega_i, \tag{7.5}$$

and

$$\vec{L}_u = \vec{L}_u(\theta_r, \phi_r), \tag{7.6}$$

where $\mathbf{F}_r$ is now the polarimetric BRDF (pBRDF). Some knowledge of the up-welled polarized radiance ($\vec{L}_u$) along the target and sensor may be gained from Rayleigh scattering theory and other sources, such as Coulson et al. (1960) and Chandrasekhar (1950). However, knowledge of the polarized downwelled radiance ($\vec{L}_d^{\Omega_i}$) is more problematic since this term often has a high spatial variability, e.g., varying cloud cover. We will take a more complete look at these terms in Sec. 7.2.

The total polarized radiance reaching a sensor aperture is then

$$\vec{L}_s = \vec{L}_r + \; \vec{L}_d + \vec{L}_u. \tag{7.7}$$

Atmospheric scattering, generally proportional to λ^{-4}, results in $\vec{L}_d$ and $\vec{L}_u$ having relatively large magnitudes at shorter wavelengths compared to $\vec{L}_r$, especially from orbital altitudes. Polarimetric remote sensing of the atmosphere uses this phenomenon to minimize ground-reflected polarization signatures to better extract atmospheric water vapor and aerosol properties [Leroy et al. (1977)].

Conversely, for polarimetric remote sensing of land features, one often wants the magnitude of the direct solar reflected radiance to be large compared to the reflected radiance from the downwelled sky and upwelled atmospheric scattering, i.e., $\vec{L}_r > \vec{L}_d, \vec{L}_u$. This provides optimal conditions for estimating the polarimetric BRDF, $\mathbf{F}_r$. Rigorously exploiting polarimetric signatures in a manner analogous to the way spectral signatures are exploited requires estimating ($\mathbf{F}_r$), given the polarized radiance reaching the aperture ($\vec{L}_s$), just as an estimate of a material's spectral reflectance factor is desired in magnitude-only remote sensing. Estimating $\mathbf{F}_r$ given the radiance at the sensor aperture proceeds as

$$\vec{L}_r \; = \; \vec{L}_s - \; \vec{L}_d - \vec{L}_u, \tag{7.8}$$

whereupon substitution and rearrangement yields

$$\tau_r \; \mathbf{F}_r \; \tau_i \; \cos\theta_i \; E_s \; = \; \vec{L}_s - \tau_r \; \iint\limits_{\Omega_i} \mathbf{F}_r \cos\theta_i \; \vec{L}_d^{\Omega_i} \; d\Omega_i - \vec{L}_u. \tag{7.9}$$

Since the exoatmospheric irradiance is randomly polarized, we can often assume that only the first column of the pBRDF Mueller matrix is of concern in the $\vec{L}_r$ expression when the specular component of reflected flux is dominated by direct solar reflection (i.e., we are viewing close to the principle plane). In

fact, overhead polarimetric remote sensing is usually restricted to the first column of the polarimetric BRDF matrix. (Solving for other matrix elements requires illumination by varying polarization states, which can be difficult if we restrict ourselves to passive sensing.) However, recognize that in certain cases the incidence flux from the sky can be highly polarized, as discussed in Sec. 7.2. With this consideration, Eq. (7.9) may be expressed as

$$\tau_r \begin{bmatrix} f_{00} \\ f_{10} \\ f_{20} \end{bmatrix} \tau_i \cos\theta_i\, E_s = \vec{L}_s - \tau_r \iint_{\Omega_i} \mathbf{F}_r \cos\theta_i\, \vec{L}_d^{\Omega_i}\, d\Omega_i - \vec{L}_u,$$

(7.10)

with the first column of the pBRDF given by

$$\begin{bmatrix} f_{00} \\ f_{10} \\ f_{20} \end{bmatrix} = \frac{\vec{L}_s - \tau_r \iint_{\Omega_i} \mathbf{F}_r \cos\theta_i\, \vec{L}_d^{\Omega_i}\, d\Omega_i' - \vec{L}_u}{\tau_r \tau_i \cos\theta_i\, E_s}.$$

(7.11)

Solving for $\mathbf{F}_r$ is complicated by its inclusion in the integral of the $\vec{L}_d$ term, which also contains the highly spatially variable and generally ill-known downwelled radiance component ($\vec{L}_d^{\Omega_i}$). However, under nominal sky conditions, the magnitude of the direct solar irradiance for $\lambda > 600$ nm is five$\times$ that of the integrated sky dome irradiance, increasing to $10\times$ for $\lambda > 1000$ nm. This makes it reasonable (particularly when operating close to the principle plane) to approximate the polarized radiance contribution of the downwelled sky radiance as an error term:

$$\begin{bmatrix} f_{00} \\ f_{10} \\ f_{20} \end{bmatrix} = \frac{\vec{L}_s - \vec{L}_u}{\tau_r \tau_i \cos\theta_i E_s} - \frac{\tau_r \iint_{\Omega_i} F_r \cos\theta_i \vec{L}_d^{\Omega_i} d\Omega_i}{\tau_r \tau_i \cos\theta_i\, E_s}$$

(7.12)

$$= \frac{\vec{L}_s - \vec{L}_u}{\tau_r \tau_i \cos\theta_i\, E_s} - \begin{bmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \end{bmatrix}.$$

(7.13)

Therefore, when operating near the principle plane, polarimetric remote sensing may recover the first column of the polarimetric BRDF Mueller matrix to within the error resulting from downwelled sky radiance, presented as

$$\begin{bmatrix} f_{00} + \epsilon_0 \\ f_{10} + \epsilon_1 \\ f_{20} + \epsilon_2 \end{bmatrix} = \frac{\vec{L}_s - \vec{L}_u}{\tau_r \tau_i \cos\theta_i\, E_s}.$$

(7.14)

Table 7.1 Maximum DoP for the Rayleigh atmo-
sphere [Coulson (1988), © A. Deepak Publishing]
and fraction of skylight to total irradiance [see Shell
and Schwarting (2004)].

$\lambda(\mu m)$	DoPmax	dsky
0.3120	0.55	—
0.3715	0.72	0.47
0.4365	0.82	0.38
0.5460	0.91	0.28
0.8090	0.97	0.18

Note that $\mathbb{C}_0$ is always positive, and while termed an "error" may be approxi-
mated with some certainty (Sec. 7.2.2), reducing the error in the retrieved pBRDF.
For diffuse surfaces, the ratio of $\mathbb{C}_0/f_{00}$ is equivalent to the ratio of the downwelled
sky irradiance to the direct solar irradiance. The linear polarization terms $\mathbb{C}_1$ and
$\mathbb{C}_2$ may be either positive or negative and represent the polarization resulting from
the downwelled sky radiance. The polarimetric downwelled radiance terms may
be estimated using radiation propagation models, as discussed in Sec. 7.2.

Situations in which the total radiance (L_s) is not dominated by the direct solar
reflectance component (L_r) should be recognized. For instance, consider the case
of viewing water at a high incidence angle outside of the principle plane. In this
situation, the dominant signal will be from the downwelled sky component (L_d),
as the highly specular water surface will be reflecting the sky radiance (L_d^Ω) in the
background. Furthermore, unlike the direct solar illumination component, the sky
illumination source may itself be appreciably polarized depending on the relative
orientation of the background sky to the sun and the spectral band (see Table 7.1).

In this section we have derived the framework for a governing equation in-
corporating the polarimetric characterization of the radiation. In order to better
understand the implications of this equation, we need to look briefly at the polar-
ized nature of the natural radiation field.

7.2 Atmospheric Scattering and the Polarized State of the Terms in the Governing Equation

In this section, we first describe the polarized nature of the radiation from the sky
(Sec. 7.2.1) and then briefly look at a validation study of a radiative transfer code
that will be used in later chapters to model the polarized behavior of the atmo-
spheric terms in the governing equation. This section draws on Shell and Schott
(2005) and Devaraj et al. (2007).

7.2.1 Characterization of the polarized state of the incident radiative field

In this section we examine in more detail the polarized nature of the atmospheric terms in the governing equation. Specifically, the downwelled and upwelled radiance components, $\overrightarrow{L_d}$ and $\overrightarrow{L_u}$, must be considered. For intensity-only remote sensing, $(\overrightarrow{L_d})$ or equivalently L_{0d} provides additional signal (reflectance) from the target. This is particularly true when the Lambertian reflectance approximation is considered. However, in polarimetric remote sensing, this term is a source of uncertainty, as the $\overrightarrow{L_d}$ polarization is dependent on the geometric location in the sky dome.

Upwelled radiance $(\overrightarrow{L_u})$ or the solar energy that is scattered in the atmosphere in the direction toward the sensor, is an additive term that increases the uncertainty in both intensity and polarimetric remote sensing. It must be subtracted from the sensor-reaching radiance when recovering the reflectance factor or the first column of the pBRDF Mueller matrix (Eq. (7.12)). For intensity-only remote sensing, it serves as a contrast reduction term. The effects of $\overrightarrow{L_u}$ in polarimetric remote sensing are more complex, since the upwelled radiance is, in general, polarized.

Conservation of energy is observed with radiance in the atmosphere, as in any other medium. Atmospheric transmittance along the incident or solar-to-target path (τ_i) and along the reflected or target-to-sensor path (τ_r) were previously introduced as factors that attenuate the energy along that path. From the conservation of energy, it is noted that

$$\tau = 1 - \rho - \alpha, \tag{7.15}$$

where ρ is reflectance or scattering and α is absorptance. τ may be expressed in terms of the *optical depth* (δ) as

$$\tau = e^{-\delta} = e^{-(\delta_s + \delta_\alpha)}, \tag{7.16}$$

where δ_s and δ_α are the optical depths for the scattered (reflected) and absorbed components. Generally, absorption results in photons being lost to thermal energy conversion by atmospheric constituents. It is the reflectance or scattering that is of most interest here, as it is these photons that are responsible for the $\overrightarrow{L_u}$ and $\overrightarrow{L_d}$ components.

Atmospheric scattering is fundamentally governed by the interaction of electromagnetic energy with molecules and particles. Solutions for the interaction of electromagnetic waves with these molecules and particles are complex and governed by Maxwell's equations. It is convenient to approximate the scattering effect based on the size of the scattering center relative to the wavelength of incident radiation. *Rayleigh scatter* results from interaction with molecules or particles that are small in comparison to λ. *Mie* or aerosol *scattering* theory applies to particles whose size is on the same order as that of λ, and finally, nonselective scattering results from particles that are large compared to λ. Using these three